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Name____________________________________

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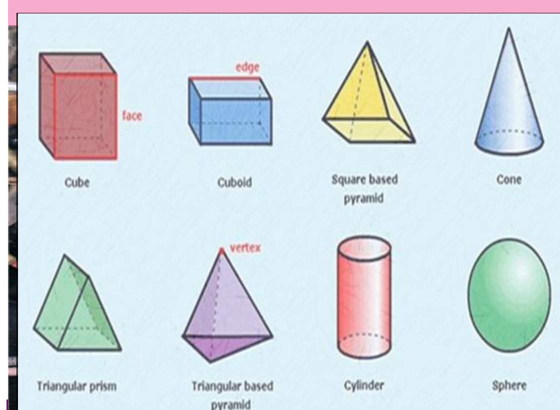
SEMISTER I | DIPLOMA IN ENGINEERING AND TECHNOLOGY

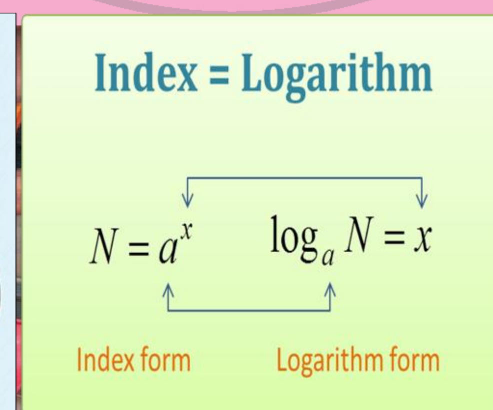
A LEARNING MANUAL

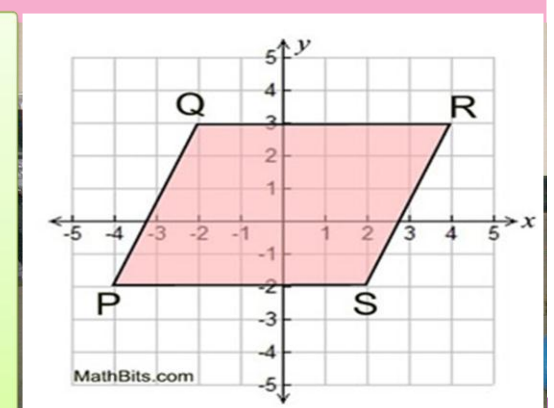
FOR

Basic Mathematics

(22103)







MAHARASHTRA STATE BOARD OF TECHNICAL EDUCATION, MUMBAI

(Autonomous) (ISO 9001 : 2015) (ISO / IEC 27001 : 2013)

A Learning Manual

for

Basic Mathematics

(22103)

Semester– (I)

(First Semester Diploma in Engineering and Technology)

Maharashtra State

Board of Technical Education, Mumbai

(Autonomous) (ISO-9001-2008) (ISO/IEC 27001:2013)

Maharashtra State
Board of Technical Education, Mumbai
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Preface

The primary focus of any engineering work in the technical education system is to develop the much needed industry relevant competencies and skills. With this in view, MSBTE embarked on this innovative 'I' Scheme curricula for engineering diploma programmes with outcome-based education.

This Learning manual is designed to help all stakeholders, especially the students, teachers and instructors to develop in the student the pre-determined outcomes. The manual begins by identifying the competency, course outcomes,. The students will become aware about the concepts of Basic Mathematics.

This manual also provides guidelines to teachers to effectively facilitate student-centered activities through each chapter by arranging and managing necessary resources ensuring the achievement of outcomes in the students.

Mathematics is the core course to develop the competencies of most of the technological courses. This basic course of Mathematics is being introduced as a foundation which will help in developing the competency and the requisite course outcomes in most of the engineering diploma programmes to cater to the needs of the industry and thereby enhance the employability. This course is an attempt to initiate the multi-dimensional logical thinking and reasoning capabilities. It will help to apply the principles of basic mathematics to solve related technology problems. Hence, the course provides the insight to analyze engineering problems scientifically using logarithms, determinants, matrices, trigonometry, coordinate geometry, mensuration and statistics.

The Learning manual development team wishes to thank MSBTE who took initiative in the development of curriculum re-design project and implementation and also acknowledge the contribution of individual course experts who have been involved in Learning manual as well as curriculum development (I scheme) directly or indirectly

Although all care has been taken to check for mistakes in this learning manual, yet it is impossible to claim perfection especially as this is the first edition. Any such errors and suggestions for improvement can be brought to our notice and are highly welcome.

Learning Manual Development Team

List of Content

Unit 1
Algebra

Course Outcome: Apply the concepts of algebra to solve engineering related problems.

Unit outcome:

a) Solve the given simple problem based on laws of logarithm.
b) Calculate the area of the given triangle by determinant method.
c) Solve given system of linear equations using matrix inversion method and by Cramer's rule.
d) Obtain the proper and improper partial fraction for the given simple rational function.

Introduction: Algebra is a simple language, used to create mathematical models of real world situations and to handle problems. The algebraic need of engineering and technology is to solve simple engineering problems using algebra. Some of the main topics coming algebra includes Logarithms, Determinants, Matrix and Partial fractions.

Logarithm

Significance of Logarithms: Logarithm is one of the best tools to simplify engineering problems.

Content of Logarithms:

Definition:

If $y = a^x$, $a > 0$, $a \neq 1$, $a \in R$, then x is called logarithm of y to the base a and it is written as $x = \log_a y$.

For example,

1) If $8 = 2^3$ then $3 = \log_2 8$
2) If $3^4 = 81$ then $\log_3 81 = 4$

Note: i) $a^x = y$ is called Exponential form or Index form and

$x = \log_a y$ is called Logarithmic form of the same expression.

ii) Logarithm of negative number and zero are not defined

LAWS OF LOGARITHM:

1. $\log_a (m\, n) = \log_a m + \log_a n$
2. $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

3. $\log_a (m)^n = n \log_a m$

4. $\log_n m = \dfrac{\log_a m}{\log_a n}$

Remark:

1. $a^0 = 1 \therefore \log_a 1 = 0$

2. $a^1 = a \therefore \log_a a = 1$

3. $a^{\log_a y} = y$

Solved Examples:

Evaluate the following

a) $\log_2 16$

Solution: $\log_2 16$

$= \log_2 2^4$

$= 4 \log_2 2$

$= 4(1)$

$= 4$

b) $\log_5 125$

Solution: $\log_5 125$

$= \log_5 5^3$

$= 3 \log_5 5$

$= 3(1)$

$= 3$

c) $25^{\log_5 8}$

Solution: $25^{\log_5 8} = [(5)^2]^{\log_5 8}$

$= 5^{2 \log_5 8}$

$= 5^{\log_5 8^2}$

$= 5^{\log_5 64} \ldots (a^{\log_a y} = y)$

$= 64$

Simplify the following

a) $\log_2 14 - \log_2 7$

Solution: $\log_2 14 - \log_2 7$

$= \log_2\left(\frac{14}{7}\right)$

$= \log_2 2$

= 1

b) $(\log_3 4)\times(\log_4 81)$

Solution: $(\log_3 4)\times(\log_4 81)$

$= \frac{\log 4}{\log 3}\times\frac{\log 81}{\log 4}$

$= \frac{\log 81}{\log 3}$

$= \frac{\log 3^4}{\log 3}$

$= 4\times\frac{\log 3}{\log 3}$

= 4

c) $\log\left(\frac{2}{3}\right)+\log\left(\frac{4}{5}\right)-\log\left(\frac{8}{15}\right)$

Solution: $\log\left(\frac{2}{3}\right)+\log\left(\frac{4}{5}\right)-\log\left(\frac{8}{15}\right)$

$= \log\left(\frac{2}{3}\times\frac{4}{5}\right)-\log\left(\frac{8}{15}\right)$

$= \log\left(\frac{8}{15}\right)-\log\left(\frac{8}{15}\right)$

$= \log\left(\frac{8}{15}\times\frac{15}{8}\right)$

$= \log(1)$

= 0

Find x if

i) $\log_3 27 = x$

Solution: $\log_3 27 = x$

$\therefore 3^x = 27$

$\therefore 3^x = 3^3$

$\therefore\ x = 3$

ii) $\log_3(x+6)=2$

Solution: Given $\log_3(x+6)=2$

$\therefore \quad x+6=3^2$

$\therefore \quad x=9-6$

$\therefore \quad x=3$

Exercises

1) Evaluate the following

1) $\log_2 32$ 2) $\log_{10} 1000$ 3) $\log_{81} 3$

4) $\log_4 0.25$ 5) $\log_{10} \sqrt[3]{1000}$ 6) $\log_3 243$

7) $\log_{343} 7$

2) Simplify the following:

1) $\log 5+\log 3-\log 2$

2) $\log\left(\frac{9}{14}\right)-\log\left(\frac{15}{16}\right)+\log\left(\frac{35}{24}\right)$

3) $2\log\left(\frac{16}{15}\right)+\log\left(\frac{25}{24}\right)-\log\left(\frac{32}{27}\right)$

4) $\frac{\log_4 64}{\log_9 81}$

5) $\log\left(\frac{225}{32}\right)-\log\left(\frac{25}{81}\right)+\log\left(\frac{64}{729}\right)$

6) $\frac{\log_7 25}{\log_7 5}=\frac{\log_5 8}{\log_5 2}$

7) $\log\left(\frac{450}{32}\right)+\log\left(\frac{25}{128}\right)+\log\left(\frac{64}{25}\right)+\log\left(\frac{32}{25}\right)$

8) $\log\left(\frac{145}{8}\right)-3\log\left(\frac{3}{2}\right)+\log\left(\frac{54}{29}\right)$

3) Find x if :

1) $\log_2(x-3)=3$ 2) $\log_3(x+4)=4$

3) $\log_3(x+5)=4$ 4) $\log_4(3x-5)=0$

5) $\log_2\left(\frac{1}{2}\right)=x$ 6) $\log_4 x=\frac{1}{2}$

Determinant

Significance: It is used to find area of triangle and solution of simultaneous equations used in engineering field.

Content of the Determinant:

Definition: The arrangement of numbers in equal number of rows and columns enclosed between two bars is called determinant.

It has a definite value.

Determinant of order 3:-

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \times \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \times \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \times \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - b_2c_1)$$

Solved Examples

I.Evaluate the following by expansion: -

1) $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 3 & 1 & 6 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 3 & 1 & 6 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ 1 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} + 5\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= 2(24 - 2) - 3(6 - 6) + 5(1 - 12)$$

$$= 2(22) - 3(0) + 5(-11) = 44 - 0 - 55$$

$$= -11$$

2) $\begin{vmatrix} 3 & -5 & -1 \\ 1 & 3 & 5 \\ -5 & 1 & 3 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 3 & -5 & -1 \\ 1 & 3 & 5 \\ -5 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix} - (-5)\begin{vmatrix} 1 & 5 \\ -5 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ -5 & 1 \end{vmatrix}$$

$$= 3(9 - 5) + 5(3 - (-25)) - 1(1 - (-15))$$

$$= 3(4) + 5(28) - 1(16)$$

$$= 12 + 140 - 16 = 136$$

Exercise

Evaluate the following determinant.

a) $\begin{vmatrix} 2 & 4 & 1 \\ 1 & 0 & 4 \\ 5 & -1 & 3 \end{vmatrix}$ b) $\begin{vmatrix} 9 & 12 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 3 \end{vmatrix}$ c) $\begin{vmatrix} 3 & -2 & 1 \\ 3 & -1 & -2 \\ 3 & -2 & -3 \end{vmatrix}$

d) $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & -3 \end{vmatrix}$ e) $\begin{vmatrix} 1 & 0 & 6 \\ 7 & 2 & 5 \\ 3 & 4 & 6 \end{vmatrix}$ f) $\begin{vmatrix} 6 & 9 & 12 \\ 2 & 3 & 4 \\ 5 & 9 & 13 \end{vmatrix}$

Cramer's Rule:-

It is the method of solving simultaneous equations using determinants.

It is also called determinant method.

Suppose $a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

are the simultaneous equations.

Then solution is $x = \frac{Dx}{D}$, $y = \frac{Dy}{D}$, $z = \frac{Dz}{D}$, provided $D \neq 0$.

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, a determinant of coefficients of x, y and z

$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, a determinant obtained from D on replacing a_1, a_2, a_3 by d_1, d_2, d_3.

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$,a determinant obtained from D on replacing b_1, b_2, b_3 by d_1, d_2, d_3.

$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$,a determinant obtained from D on replacing c_1, c_2, c_3 by d_1, d_2, d_3.

Note: Before applying Cramer’s rule, the constant terms in the equations are taken on the right hand side of the equations.

Solved Examples:-

1. Solve $x + y + z = 6$; $2x + y - 4z = -2$; $x + y - 3z = -6$

Solution: $Given \quad x + y + z = 6$;

$$2x + y - 4z = -2;$$

$$x + y - 3z = -6$$

$$\therefore D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -4 \\ 1 & 1 & -3 \end{vmatrix} = 1\{-3+4\} - 1\{-6+4\} + 1\{2-1\}$$

$$= 1 + 2 + 1 = 4$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ -2 & 1 & -4 \\ -6 & 1 & -3 \end{vmatrix} = 6\{-3+4\} - 1\{6-24\} + 1\{-2+6\}$$

$$= 6 + 18 + 4 = 28$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & -2 & -4 \\ 1 & -6 & -3 \end{vmatrix} = 1\{6-24\} - 6\{-6+4\} + 1\{-12+2\}$$

$$= -18 + 12 - 10 = -16$$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & -2 \\ 1 & 1 & -6 \end{vmatrix} = 1\{-6+2\} - 1\{-12+2\} + 6\{2-1\}$$

$$= -4 + 10 + 6 = 12$$

$$\therefore \ x = \frac{D_x}{D} = \frac{28}{4} = 7 \ ;$$

$$y = \frac{D_y}{D} = \frac{-16}{4} = -4 \ ;$$

$$z = \frac{D_z}{D} = \frac{12}{4} = 3$$

2) Following equations are obtained as a result of an experiment. ; $p_1 + p_2 - p_3 = 0$; $2p_1 + p_2 + p_3 = 26$; $p_2 + p_3 = 14$ find p_1 , p_2 and p_3 by using Cramer's rule.

Solution: Given equation can be written as

$$p_1 + p_2 - p_3 = 0$$

$$2p_1 + p_2 + p_3 = 26$$

$$0p_1 + p_2 + p_3 = 14$$

$$\text{Here } D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(1-1) - 1(2-0) - 1(2-0)$$

$$= 0 - 2 - 2$$

$$D = -4$$

$$D_{p_1} = \begin{vmatrix} 0 & 1 & -1 \\ 26 & 1 & 1 \\ 14 & 1 & 1 \end{vmatrix}$$

$$= 0(1-1) - 1(26-14) - 1(26-14)$$

$$= -12 - 12$$

$$D_{p_1} = -24$$

$$D_{p_2} = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 26 & 1 \\ 0 & 14 & 1 \end{vmatrix}$$

$$= 1(26-14) - 0(2-0) - 1(28-0)$$

$$= 12 - 28$$

$$D_{p_2} = -16$$

$$D_{p_3} = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 26 \\ 0 & 1 & 14 \end{vmatrix}$$

$$= 1(14-26) - 1(28-0) + 0(2-0)$$

$$= -12 - 28$$

$$D_{p_3} = -40$$

$$\therefore\ p_1 = \frac{D_{p_1}}{D} = \frac{-24}{-4} = 6$$

$$p_2 = \frac{D_{p_2}}{D} = \frac{-16}{-4} = 4$$

$$p_3 = \frac{D_{p_3}}{D} = \frac{-40}{-4} = 10$$

3) The voltage in an electric circuit is related by following equations:

$V_1 + V_2 + V_3 = 9$, $V_1 - V_2 + V_3 = 3$, $V_1 + V_2 - V_3 = 1$. Find V_1, V_2 and V_3 using Cramer's rule.

Solution: The equations are

$$V_1 + V_2 + V_3 = 9$$

$$V_1 - V_2 + V_3 = 3$$

$$V_1 + V_2 - V_3 = 1$$

$$\text{Here } D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 1(1-1) - 1(-1-1) + 1(1-(-1))$$

$$= 0 + 2 + 2$$

$$D = 4$$

$$D_{V_1} = \begin{vmatrix} 9 & 1 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 9(1-1) - 1(-3-1) + 1(3-(-1))$$

$$= 0 + 4 + 4$$

$$D_{V_1} = 8$$

$$D_{V_2} = \begin{vmatrix} 1 & 9 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 1(-3-1) - 9(-1-1) + 1(1-3)$$

$$= -4 + 18 - 2$$

$$D_{V_2} = 12$$

$$D_{V_3} = \begin{vmatrix} 1 & 1 & 9 \\ 1 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(-1-3) - 1(1-3) + 9(1-(-1))$$

$$= -4 + 2 + 18$$

$$D_{V_3} = 16$$

$\therefore \quad V_1 = \dfrac{D_{V_1}}{D} = \dfrac{8}{4} = 2$

$V_2 = \dfrac{D_{V_2}}{D} = \dfrac{12}{4} = 3$

$V_3 = \dfrac{D_{V_3}}{D} = \dfrac{16}{4} = 4$

Exercise

Solve the following equations using Cramer's rule

1. $x + y + z = 6$; $2x + y - 2z = -2$; $x + y - 3z + 6 = 0$
2. $x + z = 0$, $2x + 3y + 3z = 5$, $x + 3y = 5$
3. $3x + 3y - z = 11$, $2x - y + 2z = 9$, $4x + 3y + 2z = 25$
4. $8x + 3y = 2$, $y + 3z = 7$, $2x + 2z = 8$
5. $x + y + z = 3$, $x - y + z = 1$, $x + y - 2z = 0$
6. $x - y - 2z = 1$, $2x + 3y + 4z = 4$, $3x - 2y - 6z = 5$
7. Following equation are obtained as a result of experiment

$2I_1 - I_2 + I_3 = 0$, $4I_1 - I_3 = 2$, $2I_2 + I_3 = 2$. Find the values of I_1 , I_2 and I_3 by Cramer's rule

8. $4r - 7s + 2t = 4$; $3r + 6s - 7t = 5$; $2r + 4s - 2t = -3$
9. $2x_1 + 3x_2 = 5$; $x_2 - 3x_3 + 2 = 0$; $x_3 + 3x_1 = 4$

Area of a triangle:

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by,

$$\text{Area of } \Delta(ABC) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Solved Examples: -

1) Find the area of triangle whose vertices are

$(-8, -2)$, $(-4, -6)$ and $(-1, 5)$

Solution:

$$\text{Area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\text{Area of } \Delta = \frac{1}{2}\begin{vmatrix} -8 & -2 & 1 \\ -4 & -6 & 1 \\ -1 & 5 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-8(-6-5) + 2(-4-(-1)) + 1(-20-6)]$$

$$= \frac{1}{2}[88 - 6 - 26]$$

$$= 28 \text{ Sq. units}$$

2) Find the area of triangle with vertices (4, 7), (1, 3) and (5, 1).

Solution:

$$\text{Area of } \Delta\ ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 4 & 7 & 1 \\ 1 & 3 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[4(3-1) - 7(1-5) + 1(1-15)]$$

$$= \frac{1}{2}[8 + 28 - 14]$$

$$= \frac{1}{2}[22]$$

$$= 11 \text{ Sq. units}$$

Exercise:

1. Find the area of the triangle with vertices (3,1) , (-1,3) , (-3,-2)
2. Find the area of the triangle with vertices (3, 4) , (5,7) , (-2,-3)
3. Find the area of triangle with vertices A(2,1), B(1, 4) and C(– 3, 2).
4. Find the area of the triangle whose are (– 1, 5), (3, 1) and (5, 7).
5. Find the area of the triangle whose vertices are (3, 1), (– 1, 3) and (– 3, – 2).
6. Find the area of triangle whose vertices are (2, 3), (5, 7) and (– 3, 4)
7. Find the area of triangle ABC where, A ≡ (1, 2), B ≡ (– 6, 1) and C≡ (0, 8).

Matrices-

Significance of Matrices: Matrices can be used to compactly write and work with multiple linear equations, referred as system of linear equations, simultaneously

Definition:-

A set of $m \times n$ numbers arranged in a rectangular form of m rows & n columns enclosed between a pair of square brackets is called a matrix of order $m \times n$ (read as m by n).

Matrices are generally denoted by capital alphabets & its elements are denoted by small alphabets.

For e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$; $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}_{m\times n}$

In short, $A = [a_{ij}]_{m\times n}$ where

$i = No. of\ rows$ 1,2,3, … . , m &

$j = No. of\ columns$ 1,2,3, … . . . , n.

Order of a matrix:-

The order of a matrix is defined as $m \times n$ if it contains m rows & n columns.

Examples:

1. $A = [2 \quad 3 \quad -1]$ Order of A is 1×3
2. $B = \begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix}$ Order of B is 3×2
3. $C = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 9 \end{bmatrix}$ Order of C is 2×3
4. $D = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ Order of D is 2×1

Types of matrices:-

1. Row matrix: Matrix having only one row is called row matrix.
   For e.g. : $A = [2 \quad 3 \quad -1]$.
2. Column matrix: Matrix having only one column is called column matrix.
   For e.g. : $D = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$.
3. Square matrix : Matrix having equal number of rows & columns is called square matrix
   For e.g. $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix}$
   Note: In matrix A, elements 2, 3, 4 are diagonal elements and remaining are non-diagonal elements.
4. Diagonal matrix: A square matrix where all non-diagonal elements are zero is called a diagonal matrix. For e.g. : $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

5. Scalar matrix: A diagonal matrix where all diagonal elements are equal is called a scalar matrix. For e.g. : $K = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
6. Identity matrix OR Unit matrix: A scalar matrix where all diagonal elements are one (unit) is called an identity matrix or unit matrix denoted by I.
   For e.g. : $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
7. Zero matrix: A matrix having all elements equal to zero is called zero matrix.
   For e.g. : $A = [0 \quad 0 \quad 0]$; $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Algebra of matrices:

1. Addition of matrices: If two matrices are of same order then A+B can be obtained by adding the corresponding elements. Order of matrix $A + B$ is same as that of A and B.
   For e.g. if $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & 3 \\ -3 & 1 & -2 \end{bmatrix}$
   then $A + B = \begin{bmatrix} 5+4 & 6+2 & 1+3 \\ 0-3 & 2+1 & 9-2 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 4 \\ -3 & 3 & 7 \end{bmatrix}$
2. Subtraction of matrices: If two matrices are of same order then matrix $A - B$ can be obtained by subtracting the corresponding elements. Order of matrix $A - B$ is same as that of A and B.
   For e.g. if $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & 3 \\ -3 & 1 & -2 \end{bmatrix}$
   then $A - B = \begin{bmatrix} 5-4 & 6-2 & 1-3 \\ 0+3 & 2-1 & 9+2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 \\ 3 & 1 & 11 \end{bmatrix}$
3. Scalar Multiplication: If A is a matrix and ‘k’ is a scalar then the matrix ‘kA’ is obtained by multiplying every element of the matrix A by ‘k’.
   For e.g. if $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 9 \end{bmatrix}$ $then\ 5A = \begin{bmatrix} 25 & 30 & 5 \\ 0 & 10 & 45 \end{bmatrix}$ where k=5

Solved examples:

1. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 6 & -1 \\ 0 & 3 \end{bmatrix}$ find $2A + 3B$.

   Solution: $2A + 3B = 2\begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 6 & -1 \\ 0 & 3 \end{bmatrix}$

   $= \begin{bmatrix} 4 & -6 \\ 8 & 0 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 18 & -3 \\ 0 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 4+3 & -6+6 \\ 8+18 & 0-3 \\ -2+0 & -4+9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 26 & -3 \\ -2 & 5 \end{bmatrix}$$

2. If A = $\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$, C = $\begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$ then find 5A – 3B + 2C

Solution: 5A – 3B + 2C

$$=5\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 25 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 12 & -3 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 14 \\ 10 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10-12+2 & 25+3+14 \\ 0-6+10 & 5-0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 42 \\ 4 & 9 \end{bmatrix}$$

3. Find the value of x and y satisfying the equation

$$\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

Solution: $\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1+3 & x+1 & 0+2 \\ y+4 & 2+3 & 4-2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 & x+1 & 2 \\ y+4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

By using equality of matrices, $x+1=2$ and $y+4=6$

$$\therefore x = 1 \quad \& \quad y = 2$$

4. If A = $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, find the matrix 'X' such that 2A + X = 3B

Solution: 2A + X = 3B

$\therefore$ X= 3B – 2A

$$\therefore X = 3\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -6 \\ -3 & 12 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4 & -6+2 \\ -3-8 & 12-6 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 & -4 \\ -11 & 6 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$

Then prove that $(A+B)+C = A+(B+C)$

Solution: $L.H.S. = (A+B)+C$

$$= \left(\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}\right) + \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 11 \\ 7 & 3 \end{bmatrix}$$

$R.H.S. = A+(B+C)$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} + \left(\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 11 \\ 7 & 3 \end{bmatrix}$$

$\therefore L.H.S. = R.H.S.$

Exercise:

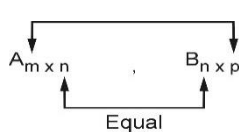
1) If $A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, find $2A - 3B$
2) If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$, $Z = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$. Show that $3X + Y = Z$
3) If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, find $2A + 3B - 4I$
4) If $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 3 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix}$, find $3A + 4B - 2C$
5) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
   Verify that $(A + B) + C = A + (B + C)$
6) If $A = \begin{bmatrix} x & 2 & -5 \\ 3 & 1 & 2y \end{bmatrix}$, $B = \begin{bmatrix} 2y+5 & 6 & -15 \\ 9 & 3 & -6 \end{bmatrix}$ and if $3A = B$, find x and y
7) If $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 4 & -2 \end{bmatrix}$, verify that $A + B = B + A$

Matrix multiplication:

The product of two matrices A and B is possible only if the number of columns in A is equal to the number of rows in B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix

$B = [b_{ij}]$ be an $n \times p$ matrix.



Order of A× B is m×p

Method of Multiplication of two matrices:

$$\text{Let } A = \begin{matrix} R_1 \longrightarrow \\ R_2 \longrightarrow \end{matrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} p & q & r \\ x & y & z \end{bmatrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{matrix}$$

$$\text{then } AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} = \begin{bmatrix} \text{ap} + \text{bx} & \text{aq} + \text{by} & \text{ar} + \text{bz} \\ \text{cp} + \text{dx} & \text{cq} + \text{dy} & \text{cr} + \text{dz} \end{bmatrix}$$

Note: R_1C_1 means multiplying the elements of first row of A with corresponding elements of first column of B.

Note: In matrices, matrix multiplication is not commutative.

i.e. $A \times B \neq B \times A$ in general

Solved examples:-

1. If $A = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 0 & 2 \end{bmatrix}$ find matrix AB

Solution : $AB = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 0 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} (3)(2)+(4)(3)+(-2)(0) & (3)(-1)+(4)(4)+(-2)(2) \\ (2)(2)+(1)(3)+(0)(0) & (2)(-1)+(1)(4)+(0)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 6+12+0 & -3+16-4 \\ 4+3+0 & -2+4+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18 & 9 \\ 7 & 2 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix}$ $then\, find\ A^2 - 3\text{I}$.

Solution : $A^2 = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 4-1+0 & -2-3+0 & 0+4+0 \\ 2+3-20 & -1+9+12 & 0-12-16 \\ 10-3+20 & -5-9-12 & 0+12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 & 4 \\ -15 & 20 & -28 \\ 27 & -26 & 28 \end{bmatrix}$$

$$3I = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore A^2 - 3\mathrm{I} = \begin{bmatrix} 3 & -5 & 4 \\ -15 & 20 & -28 \\ 27 & -26 & 28 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore A^2 - 3\mathrm{I} = \begin{bmatrix} 0 & -5 & 4 \\ -15 & 17 & -28 \\ 27 & -26 & 25 \end{bmatrix}$$

3. Find x and y if

$$\left\{ 4\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 & -1 \\ 2 & -3 & 4 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Solution:

$$\left\{ 4\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 & -1 \\ 2 & -3 & 4 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 4 & 8 & 0 \\ 8 & -4 & 12 \end{bmatrix} - \begin{bmatrix} 2 & 6 & -2 \\ 4 & -6 & 8 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 4-2 & 8-6 & 0+2 \\ 8-4 & -4+6 & 12-8 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 4+0-2 \\ 8+0-4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

x=2 and y =4

4. Find x, y, z if $\left\{ \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 3 & 0 & 2 \\ 1 & 4 & 5 \\ 2 & 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Solution:

$$\left\{ \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 4 \\ 2 & 8 & 10 \\ 4 & 2 & 0 \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1+6 & 3+0 & 2+4 \\ 2+2 & 0+8 & 1+10 \\ 3+4 & 1+2 & 2+0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 7 & 3 & 6 \\ 4 & 8 & 11 \\ 7 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 7+6+18 \\ 4+16+33 \\ 7+6+6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 31 \\ 53 \\ 19 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore x = 31$$

$$y = 53$$

$$z = 19$$

5. *If* $A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$ *show that* $A^2 - 8A$ *is scalar matrix*

Solution : $A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix} . \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+16+16 & (8(4)+8+16 & 8+16+8 \\ 8+8+16 & 16+4+16 & 16+8+8 \\ 8+16+8 & 16+8+8 & 16+16+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix}$$

$$A^2 - 8A = \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} - 8 \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} - \begin{bmatrix} 16 & 32 & 32 \\ 32 & 16 & 32 \\ 32 & 32 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 36-16 & 32-32 & 32-32 \\ 32-32 & 36-16 & 32-32 \\ 32-32 & 32-32 & 36-16 \end{bmatrix}$$

$$A^2 - 8A = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} \text{ is a scalar matrix}$$

Exercise :

1) If $A = \begin{bmatrix} 1 & -5 \\ 6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, find the matrix AB – 2I
2) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}$, find AB
3) If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find $A^2 - 9A + 14I$, where I is unit matrix.
4) If $\left\{ 3\begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 3 & -3 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 \\ -2 & 3 \\ -5 & 4 \end{bmatrix} \right\} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find x, y, z
5) Find x and y satisfying the matrix equation
$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x & y & 3 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 7 \\ 9 & 4 & 13 \end{bmatrix}$
6) If $A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$, find A^2
7) If $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ prove that $A^2 = I$

Transpose of a matrix:

Definition: The transpose of a matrix A is obtained by interchanging rows and columns of matrix A. It is denoted by A' or A^t or A^T

For e.g.: If $A = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$ $then\ A' = \begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix}$

Properties:

I. $(A')' = A$
II. $(A + B)' = A' + B'$

III. $(A \times B)' = B' \times A'$

Symmetric Matrix:

Definition: In a matrix A, if $a_{ij} = a_{ji}$ for all $i\ and\ j$ then matrix is known as symmetric matrix i.e. if $A = A'$ then matrix is known as symmetric matrix.

For e.g. $A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & 5 & 3 \\ -4 & 3 & 9 \end{bmatrix}$

Skew Symmetric Matrix:

Definition: In a matrix A, if $a_{ij} = -a_{ji}$ for all $i\ and\ j$ then matrix is known as skew symmetric matrix i.e. if $A = -A'$ then matrix is skew symmetric matrix.

For e.g. $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

$If\ AA' = A'A = I$ then A is called **orthogonal matrix.**

Solved examples:

1. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$, verify that $(A + B)^T = A^T + B^T$

Solution:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ -1 & 0 \end{bmatrix} \qquad B^T = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 3+2 & -1+4 \\ 4+1 & 5+3 & 0+0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 8 & 0 \end{bmatrix}$$

$$\therefore (A + B)^T = \begin{bmatrix} 1 & 5 \\ 5 & 8 \\ 3 & 0 \end{bmatrix} \qquad ...(1)$$

$$A^T + B^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4+1 \\ 3+2 & 5+3 \\ -1+4 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 5 & 8 \\ 3 & 0 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2)

$$(A+B)^T = A^T + B^T$$

2. If $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ then verify that $(AB)' = B'A'$

Solution:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-3 & -2+0 & 4-3 \\ 3+5 & -1+0 & 2+5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & -2 & 1 \\ 8 & -1 & 7 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 3 & 8 \\ -2 & -1 \\ 1 & 7 \end{bmatrix} \quad \ldots (i)$$

$$B'\ A' = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 6-3 & 3+5 \\ -2+0 & -1+0 \\ 4-3 & 2+5 \end{bmatrix}$$

$$B'\ A' = \begin{bmatrix} 3 & 8 \\ -2 & -1 \\ 1 & 7 \end{bmatrix} \quad \ldots (ii)$$

From (i) and (ii) $\quad (AB)' = B' \cdot A'$

3. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'\ A'$

Solution:

Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$$A' = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \text{ and } B' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 0+2-1 & 0+0-3 \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix} \quad \ldots \text{(i)}$$

$$B' \cdot A' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+5+0 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{bmatrix}$$

$$B' \cdot A' = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix} \quad \ldots \text{(ii)}$$

From (i) and (ii)

$(AB)' = B' \cdot A'$

Exercise:

1) If $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 \\ -3 & 4 \end{bmatrix}$, find $(AB)^T$
2) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, verify that $(A+B)^T = A^T + B^T$
3) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, verify that $(AB)^T = B^T \cdot A^T$
4) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$, verify that $(AB)' = B' \cdot A'$
5) If $A = \begin{bmatrix} 1 & -3 \\ -2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$, verify that $(AB)' = B' \cdot A'$
6) If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 7 \\ -5 & 6 \\ -4 & 4 \end{bmatrix}$ then show that $(AB)' = B' \cdot A'$

Singular matrix

A square matrix A is called singular matrix if det (A) or $|A| = 0$.

Non-Singular matrix

A square matrix A is called non-singular, if det (A) or $|A| \neq 0$.

Solved Example:

1) If $A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$ Show that the matrix AB is non-singular.

Solution: Given $A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0+0+1 & -2+0+1 \\ 0+4+3 & 1+6+3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 7 & 10 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 1 & -1 \\ 7 & 10 \end{vmatrix} = 10 + 7$$

$$|AB| = 17 \neq 0$$

$\therefore$ AB is a non-singular matrix.

Exercise:

1) Prove that the matrix $\begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$ is nonsingular matrix.
2) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ Show that AB is non-singular matrix.
3) If $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$ decide whether AB is singular or non-singular matrix ?

Adjoint of a matrix:

Adjoint of a matrix is the transpose of co-factor matrix

$$\therefore \text{Adj } A = [c_{ij}]^t$$

Co-factor matrix is a matrix of co-factors=$[c_{ij}]$

$where\ c_{ij} = (-1)^{i+j} \times M_{ij}$ where

Minor M_{ij} = determinant of matrix obtained by deleting i^{th} row & j^{th} column of given matrix.

Solved examples:

1. If A=$\begin{bmatrix} -1 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, find Adj A

Solution: Given A= $\begin{bmatrix} -1 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$$c_{11} = (-1)^{1+1} \times \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = +(4-4) = 0$$

$$c_{12} = (-1)^{1+2} \times \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -(2-6) = 4$$

$$c_{13} = (-1)^{1+3} \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = +(4-12) = -8$$

$$c_{21} = (-1)^{2+1} \times \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = 1$$

$$c_{22} = (-1)^{2+2} \times \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = +(-1-3) = -4$$

$$c_{23} = (-1)^{2+3} \times \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -(-2-3) = 5$$

$$c_{31} = (-1)^{3+1} \times \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} = +(2-4) = -2$$

$$c_{32} = (-1)^{3+2} \times \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = -(-1-3) = 4$$

$$c_{33} = (-1)^{3+3} \times \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} = +(-4-2) = -6$$

$$\therefore Matrix\ of\ cofactors = C = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -4 & 5 \\ -2 & 4 & -6 \end{bmatrix}$$

$$\therefore Adj\ A = C^t = \begin{bmatrix} 0 & 1 & -2 \\ 4 & -4 & 4 \\ -8 & 5 & -6 \end{bmatrix}$$

Exercise:

1) Find adjoint of matrix of A. If A = $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$
2) Find the adjoint of matrix of A if A = $\begin{bmatrix} 1 & 2 & 6 \\ 7 & 2 & 5 \\ 8 & 2 & 10 \end{bmatrix}$
3) Find adjoint of the matrix A. If A = $\begin{bmatrix} 2 & -1 & -3 \\ 3 & -4 & -2 \\ 5 & 2 & 4 \end{bmatrix}$
4) If A = $\begin{bmatrix} -1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ Find adjoint of A

Inverse of a matrix:

If matrix A is a non-singular matrix and if there exists a matrix B such that $A \times B = B \times A = I$ then matrix B is the inverse of A.

Notation: Inverse of A = A^{-1}

Formula: $A^{-1} = \frac{1}{\det A} \times \text{Adj } A$

Solved example:

1. Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution : Given $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

$$|A| = 3(3+1) - 1(12-2) + 2(-4-2)$$
$$= 12 - 10 - 12 = -10 \neq 0$$
$$\therefore A^{-1} \text{ exists}$$

$$c_{11} = (-1)^{1+1} \times \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = +(3+1) = 4$$
$$c_{12} = (-1)^{1+2} \times \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = -(12-2) = -10$$
$$c_{13} = (-1)^{1+3} \times \begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = +(-4-2) = -6$$
$$c_{21} = (-1)^{2+1} \times \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -(3+2) = -5$$
$$c_{22} = (-1)^{2+2} \times \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = +(9-4) = 5$$
$$c_{23} = (-1)^{2+3} \times \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -(-3-2) = 5$$
$$c_{31} = (-1)^{3+1} \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = +(1-2) = -1$$
$$c_{32} = (-1)^{3+2} \times \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -(3-8) = 5$$
$$c_{33} = (-1)^{3+3} \times \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = +(3-4) = -1$$

$$\therefore Matrix\ of\ cofactors = C = \begin{bmatrix} 4 & -10 & -6 \\ -5 & 5 & 5 \\ -1 & 5 & -1 \end{bmatrix}$$

$$\therefore Adj\ A = C^t = \begin{bmatrix} 4 & -5 & -1 \\ -10 & 5 & 5 \\ -6 & 5 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{-10} \times \begin{bmatrix} 4 & -5 & -1 \\ -10 & 5 & 5 \\ -6 & 5 & -1 \end{bmatrix}$$

2. Find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}$$

$$= 3 + 6 - 8$$

$$= 1 \neq 0$$

$\therefore A^{-1}$ *exists*

To find cofactor matrix

$$C_{11} = +\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = +(-3+4) = 1$$

$$C_{12} = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -(2-0) = -2$$

$$C_{13} = +\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = +(-2-0) = -2$$

$$C_{21} = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -(-3+4) = -1$$

$$C_{22} = +\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = +(3-0) = 3$$

$$C_{23} = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = -(-3-0) = 3$$

$$C_{31} = +\begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = +(-12+12) = 0$$

$$C_{32} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -(12-8) = -4$$

$$C_{33} = +\begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = +(-9+6) = -3$$

$$\text{Cofactor matrix} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$AdjA = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} AdjA = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Exercise:

1) Find the inverse of the matrix A = $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ by using adjoint matrix.
2) Find the inverse of the matrix A = $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ by using adjoint method.
3) Find A^{-1} by adjoint method if A = $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$
4) Find inverse of the matrix A = $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}$ using adjoint method.
5) Find the inverse of matrix by adjoint method A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Solution of simultaneous equations:

Suppose $a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

are the simultaneous equations.

These equations can be represented in matrix form as follows:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{i.e.} \quad A \times X = B \text{ where}$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\therefore$ *Solution is given by* $X = A^{-1} \times B$ where

$$A^{-1} = \frac{1}{\det A} \times Adj\ A$$

Solved Example:

1. Solve the equation using matrix method:

$x + y + z = 3;\ x + 2y + 3z = 4;\ x + 4y + 9z = 6$

Solution : $x + y + z = 3;$

$x + 2y + 3z = 4;$

$x + 4y + 9z = 6$

Matrix Equation: $A \times X = B$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$

$\therefore |A| = 1(18 - 12) - 1(9 - 3) + 1(4 - 2) = 6 - 6 + 2 = 2 \neq 0$

$\therefore A^{-1}$ exists

$c_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = +(18 - 12) = 6$

$c_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9 - 3) = -6$

$c_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = +(4 - 2) = 2$

$c_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9 - 4) = 5$

$c_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = +(9 - 1) = 8$

$c_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4 - 1) = -3$

$c_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = +(3 - 2) = 1$

$c_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3 - 1) = -2$

$c_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = +(2 - 1) = 1$

$\therefore Matrix\ of\ cofactors = C = \begin{bmatrix} 6 & -6 & 2 \\ 5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$

$\therefore Adj\ A = C^t = \begin{bmatrix} 6 & 5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{\det A} \times Adj\ A = \frac{1}{2} \times \begin{bmatrix} 6 & 5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

$\therefore X = A^{-1} \times \mathrm{B}$

$= \frac{1}{2} \times \begin{bmatrix} 6 & 5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$

$$= \frac{1}{2} \times \begin{bmatrix} 18 - 20 + 6 \\ -18 + 32 - 12 \\ 6 - 12 + 6 \end{bmatrix}$$

$$= \frac{1}{2} \times \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore x = 2;\ y = 1;\ z = 0$

2. Using matrix inversion method, solve the following system of equations

   x+ y + z = 3 , 3x - 2y + 3z = 4 5x + 5y +z = 11

Solution: x+ y + z = 3

3x - 2y + 3z = 4

5x + 5y +z = 11

Given system of equation can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & 3 \\ 5 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 11 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow X = A^{-1} \times B$$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ 4 \\ 11 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 3 & -2 & 3 \\ 5 & 5 & 1 \end{vmatrix}$

$$= 1[-17] - 1[-12] + 1[25]$$

$$= -17 + 12 + 25$$

$$|A| = 20 \neq 0$$

A^{-1} *exists*

To find the cofactor matrix of A

$$C_{11} = +\begin{vmatrix} -2 & 3 \\ 5 & 1 \end{vmatrix} = +(-2 - 15) = -17$$

$$C_{12} = -\begin{vmatrix} 3 & 3 \\ 5 & 1 \end{vmatrix} = -(3 - 15) = 12$$

$$C_{13} = +\begin{vmatrix} 3 & -2 \\ 5 & 5 \end{vmatrix} = +(15+10) = 25$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} = -(1-5) = 4$$

$$C_{22} = +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} = +(1-5) = -4$$

$$C_{23} = -\begin{vmatrix} 1 & 1 \\ 5 & 5 \end{vmatrix} = -(5-5) = 0$$

$$C_{31} = +\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} = +(3+2) = 5$$

$$C_{32} = -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = -(3-3) = 0$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = +(-2-3) = -5$$

$$\text{cofactor matrix} = \begin{bmatrix} -17 & 12 & 25 \\ 4 & -4 & 0 \\ 5 & 0 & -5 \end{bmatrix}$$

$$AdjA = \begin{bmatrix} -17 & 4 & 5 \\ 12 & -4 & 0 \\ 25 & 0 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} AdjA = \frac{1}{20}\begin{bmatrix} -17 & 4 & 5 \\ 12 & -4 & 0 \\ 25 & 0 & -5 \end{bmatrix}$$

$\therefore X = A^{-1} \times \text{B}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -17 & 4 & 5 \\ 12 & -4 & 0 \\ 25 & 0 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 11 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -51+16+55 \\ 36-16+0 \\ 75+0-55 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore x = 1, y = 1, z = 1$$

Exercise:

1) Solve the equations using matrix method
   x + 3y + 2z = 6, 3x – 2y + 5z = 5, 2x – 3y + 6z = 7
2) Using matrix method, solve the following equations
   x + 3y + 3z = 12; x + 4y + 4z = 15; x + 3y + 4z = 13
3) Using matrix inversion method solves the equations.
   x + y + z = 3; x + 2y + 3z = 4; x + 4y + 9z = 6
4) Using matrix method, solve the simultaneous equation.
   x + y + z = 6; x – y + 2z = 5 ; 2x + y – z = 1
5) Solve by matrix method the set of equations.
   x + y + z = 2 ; y + z = 1 ; z + x = 3
6) Solve the following equations by matrix inversion method.
   3x + y + 2z = 3 , 2x – 3y – z = –3, x + 2y + z = 4
7) Solve the equations by inversion matrix method.
   3x + y + 2z = 3, 2x – 3y – z = – 3, x + 2y + z = 4

Partial fraction

**Significance : ** Partial fraction plays a very important role in separation of given expression.

Rational fraction: An expression of the type $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x , is called rational fraction.

e.g $\frac{x^2+4x+8}{x+2}$, is a rational fraction.

There are two types of fractions proper fraction and improper fraction.

Proper fraction: In the fraction $\frac{P(x)}{Q(x)}$, if the degree of the polynomial P(x) is smaller than the degree of the polynomial Q(x) then the fraction is said to be proper fraction .

e.g $\frac{x+2}{x^3+5x+6}$ is a proper fraction.

Improper fraction : In the fraction $\frac{P(x)}{Q(x)}$, if the degree of the polynomial P(x) is greater than or equal to the degree of the polynomial Q(x) then the fraction is said to be improper fraction

e.g. $\frac{x^3-3}{x^2-4}$, $\frac{x^2+1}{x^2-1}$ are improper fractions.

Improper fraction to Proper fraction: Any improper fraction can be expressed as sum of a polynomial and a proper fraction by division method.

i.e. Improper fraction $= \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$

$= Q + \frac{R}{D}$

Partial Fraction: Every proper fraction $\frac{P(x)}{Q(x)}$ can be expressed as sum or difference of simple fractions. These simple fractions are called as partial fractions of $\frac{P(x)}{Q(x)}$

e.g. $\frac{1}{(x+2)(x+1)} = \frac{-1}{(x+2)} + \frac{1}{(x+1)}$

Here $\frac{-1}{(x+2)}, \frac{1}{(x+1)}$ is proper fraction of $\frac{1}{(x+2)(x+1)}$

Methods to find partial fractions:

Depending upon the nature of factors of the denominator there are three cases

CASE I: When denominator contains non repeated linear factors:

If the denominator contains non-repeated linear factor of the type (ax + b) then for each such factor there is partial fraction of the type $\frac{A}{(ax+b)}$

In general $\frac{ax+b}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)} + \frac{C}{(x-\gamma)}$

Where A, B, C etc. are constants to be determined.

SOLVED EXMPLES

1) Resolve into partial fractions $\frac{1}{x^2-x}$

Solution: $\frac{1}{x^2-x} = \frac{1}{x(x-1)}$

Let $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{(x-1)}$ … (1)

$$\frac{1}{x(x-1)} = \frac{A(x-1) + B(x)}{x(x-1)}$$

Comparing numerators

$\therefore\ 1 = A(x-1) + B(x)$ .....(2)

To find A, Put x = 0 in equation (2), we get

$$1 = A(0-1) + B(0)$$
$$\therefore\ 1 = A(-1)$$
$$\therefore\ 1 = -A$$
$$\therefore\ A = -1$$

To find B, Put x = 1 in equation (2), we get

$$1 = A(0) + B(1)$$
$$\therefore\ 1 = B(1)$$
$$\therefore\ B = 1$$

Putting the value of A and B in equation (1)

$$\therefore\ \frac{1}{x^2 - x} = \frac{-1}{x} + \frac{1}{(x-1)}$$

2) Resolve into partial fractions $\frac{x^2+1}{x(x^2-1)}$

Solution: $\frac{x^2+1}{x(x^2-1)} = \frac{x^2+1}{x(x+1)(x-1)}$

Let $\frac{x^2+1}{x(x+1)(x-1)} = \frac{A}{X} + \frac{B}{X+1} + \frac{C}{X-1}$ ... (1)

$$\therefore\ \frac{x^2+1}{x(x+1)(x-1)} = \frac{A(x+1)(x-1) + B(x)(x-1) + C(x)(x+1)}{x(x+1)(x-1)}$$

Comparing numerators

$\therefore\ x^2 + 1 = A(x+1)(x-1) + B(x)(x-1) + C(x)(x+1)$... (2)

To find A, Put x = 0 in equation (2), we get

$$0^2 + 1 = A(0+1)(0-1) + B(0) + C(0)$$
$$1 = A(1)(-1)$$
$$\therefore\ 1 = A(-1)$$
$$\therefore\ 1 = -A$$
$$\therefore\ A = -1$$

To find B, Put x = –1 in equation (2), we get

$$(-1)^2 + 1 = A(0) + B(-1)(-1-1) + C(0)$$
$$\therefore\ 1 + 1 = B(-1)(-2)$$
$$\therefore\ 2 = B(2)$$
$$\therefore\ B = 1$$

To find C, Put x = 1 in equation (2), we get

$$(1)^2 + 1 = A(0) + B(0) + C(1)(1+1)$$

$$1 + 1 = C(1)(2)$$
$$2 = C(2)$$
$$\therefore \; C = 1$$

Putting the value of A, B, C in equation (1)

$$\frac{x^2 + 1}{x(x^2 - 1)} = \frac{-1}{x} + \frac{1}{(x + 1)} + \frac{1}{(x - 1)}$$

3) Resolve into partial fractions $\frac{x + 3}{(x - 1)(x + 1)(x + 5)}$

Solution : Let,

$$\frac{x + 3}{(x - 1)(x + 1)(x + 5)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 5}$$

$$\therefore \; x + 3 = A(x + 1)(x + 5) + B(x - 1)(x + 5) + C(x - 1)(x + 1)$$

Put $x = 1$

$$\therefore \; 4 = A(2)(6)$$
$$\therefore \; 4 = 12A$$
$$\therefore \; A = \frac{1}{3}$$

Put $x = -1$

$$-1 + 3 = B(-2)(4)$$
$$\therefore \; 2 = -8B$$
$$\therefore B = -\frac{1}{4}$$

Put $x = -5$

$$-5 + 3 = C(-6)(-4)$$
$$\therefore -2 = 24C$$
$$\therefore \; C = \frac{-1}{12}$$

$$\frac{x + 3}{(x - 1)(x + 1)(x + 5)} = \frac{\frac{1}{3}}{x - 1} + \frac{-\frac{1}{4}}{x + 1} + \frac{-\frac{1}{12}}{x + 5}$$

EXERCISE:

Resolve into partial fractions

1) $\frac{1}{x^2 - 1}$ 2) $\frac{1}{x^2 - x}$ 3) $\frac{1}{x^2 + 3x + 2}$

4) $\dfrac{2x + 3}{x^2 - 2x - 3}$ 5) $\dfrac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}$ 6) $\dfrac{2x - 3}{(x^2 - 1)(2x + 3)}$

7) $\dfrac{3x - 1}{(x - 4)(2x + 1)(x - 1)}$ 8) $\dfrac{x^2 + 5x + 7}{(x - 1)(x + 2)(x + 4)}$ 9) $\dfrac{x + 4}{x(x + 1)(x + 2)}$

Problem Reducible to Case I after Suitable Substitution

1. Resolve into partial fractions $\dfrac{\tan\theta + 1}{(\tan\theta + 2)(\tan\theta + 3)}$

Solution: Given $\dfrac{\tan\theta + 1}{(\tan\theta + 2)(\tan\theta + 3)}$

Put $\tan\theta = t$ then $\dfrac{t + 1}{(t + 2)(t + 3)}$

Now $\dfrac{t + 1}{(t + 2)(t + 3)}$ is a proper fraction. Here factors of denominator are linear and unequal.

Let, $$\frac{t + 1}{(t + 2)(t + 3)} = \frac{A}{(t + 2)} + \frac{B}{(t + 3)} \quad \ldots (1)$$

$$\therefore \quad \frac{t + 1}{(t + 2)(t + 3)} = \frac{A(t + 3) + B(t + 2)}{(t + 2)(t + 3)}$$

Comparing numerators,

$$t + 1 = A(t + 3) + B(t + 2) \quad \ldots (2)$$

To find A, Put t = – 2 in equation (2), we get

$$-2 + 1 = A(-2 + 3) + B(0)$$

$$\therefore \quad -1 = A(1)$$

$$\therefore \quad -1 = A$$

$$\therefore \quad A = -1$$

To find B, Put t = – 3 in equation (2), we get

$$-3 + 1 = A(0) + B(-3 + 2)$$

$$\therefore \quad -2 = B(-1)$$

$$\therefore \quad -2 = -B$$

$$\therefore \quad B = 2$$

Putting the value of A and B in equation (1)

$$\frac{t + 1}{(t + 2)(t + 3)} = \frac{-1}{(t + 2)} + \frac{2}{(t + 3)}$$

But $t = \tan\theta$

$$\frac{\tan\theta + 1}{(\tan\theta + 2)(\tan\theta + 3)} = \frac{-1}{(\tan\theta + 2)} + \frac{2}{(\tan\theta + 3)}$$

2. Resolve into partial fractions $\dfrac{e^x}{e^{2x} + 4e^x + 3}$

Solution: Given $\dfrac{e^x}{e^{2x} + 4e^x + 3}$ $[e^{2x} = (e^x)^2 = t^2]$

Put $e^x = t$ then $\dfrac{t}{t^2 + 4t + 3}$

Now $\dfrac{t}{t^2 + 4t + 3}$ is a proper fraction.

$$\frac{t}{t^2 + 4t + 3} = \frac{t}{(t + 3)(t + 1)}$$

Let, $$\frac{t}{(t + 3)(t + 1)} = \frac{A}{(t + 3)} + \frac{B}{(t + 1)} \quad \ldots (1)$$

$$\therefore \quad \frac{t}{(t + 3)(t + 1)} = \frac{A(t + 1) + B(t + 3)}{(t + 3)(t + 1)}$$

Comparing numerators,

$$\therefore \quad t = A(t + 1) + B(t + 3) \quad \ldots (2)$$

To find A, Put $t = -3$ in equation (2), we get

$$-3 = A(-3 + 1) + B(0)$$

$$\therefore \quad -3 = A(-2)$$

$$\therefore \quad -3 = -2A$$

$$\therefore \quad 3 = 2A$$

$$\therefore \quad A = \frac{3}{2}$$

To find B, Put $t = -1$ in equation (2), we get

$$-1 = A(0) + B(-1 + 3)$$

$$\therefore \quad -1 = B(2)$$

$$\therefore \quad B = \frac{-1}{2}$$

Putting the value of A and B in equation (1)

$$\frac{t}{(t + 3)(t + 1)} = \frac{3/2}{(t + 3)} + \frac{-1/2}{(t + 1)}$$

But $t = e^x$

$$\therefore \quad \frac{e^x}{(e^x + 3)(e^x + 1)} = \frac{3}{2(e^x + 3)} - \frac{1}{2(e^x + 1)}$$

Exercise:

Resolve into partial fractions

1. $\dfrac{\sin\theta + 1}{(\sin\theta + 2)(\sin\theta + 3)}$

2. $\dfrac{x^2 + 1}{2x^4 + 5x^2 + 2}$

3. $\dfrac{e^x + 1}{(e^x + 2)(e^x + 3)}$ 4. $\dfrac{\tan\theta}{(\tan\theta + 2)(\tan\theta + 3)}$ 5. $\dfrac{\log x}{(\log x - 2)(\log x - 3)}$

CASE II: When denominator contains repeated linear factors:

$$\frac{ax+b}{(x-\alpha)(x-\beta)^2} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)} + \frac{C}{(x-\beta)^2}$$

Here $(x-\alpha)$ is linear non- repeated factor and $(x-\beta)^2$ is repeated linear factor.

SOLVED EXAMPLES:

1. Resolve into partial fractions $\dfrac{9}{(x-1)(x+2)^2}$

Solution: $\dfrac{9}{(x-1)(x+2)^2} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x+2)^2}$ … (1)

$$\frac{9}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

Comparing numerators

$\therefore$ $9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$ … (2)

To find A, Put x = 1 in equation (2), we get

$9 = A(1+2)^2 + B(0) + C(0)$

$\therefore$ $9 = A(3)^2$

$\therefore$ $9 = A(9)$

$\therefore$ $A = \dfrac{9}{9}$ $\therefore$ A = 1

To find C, Put x = –2 in equation (2), we get

$9 = A(0) + B(0) + C(-2-1)$

$\therefore$ $9 = C(-3)$

$\therefore$ $C = \dfrac{-9}{3}$ $\therefore$ C = – 3

To find B, Put x = 0 in equation (2), we get

$9 = A(0+2)^2 + B(0-1)(0+2) + C(0-1)$

$\therefore$ $9 = A(2)^2 + B(-1)(2) + C(-1)$

$\therefore$ $9 = A(4) + B(-2) + C(-1)$

$\therefore$ $9 = 1(4) + B(-2) - 3(-1)$

$\therefore$ $9 = 4 + B(-2) + 3$

$\therefore$ $9 = 7 + B(-2)$

$\therefore$ $9 - 7 = B(-2)$

$\therefore$ $2 = B(-2)$

$\therefore \quad B = \frac{-2}{2} \quad \therefore B = -1$

Putting the value of A, B, C in equation (1), we get

$$\frac{9}{(x-1)(x+2)^2} = \frac{1}{(x-1)} - \frac{1}{(x+2)} - \frac{3}{(x+2)^2}$$

2 Resolve into partial fractions $\frac{2x+1}{x^2(x+1)}$

Solution: $\frac{2x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} ------------1$

$$\frac{2x+1}{x^2(x+1)} = \frac{x(x+1)A+(x+1)B+x^2C}{x^2(x+1)}$$

$$\therefore \quad 2x+1 = x(x+1)A+(x+1)B+x^2C ----------2$$

Put x = 0 in equation 2 , we get

$$2(0)+1 = 0(0+1)A+(0+1)B+(0)^2C$$

$$1 = 0(1)A+(1)B+(0)^2C$$

$$1 = 0A+(1)B+0C$$

$$1 = B$$

Put x = – 1 in equation 2 , we get

$$2(-1)+1 = -1(-1+1)A+(-1+1)B+(-1)^2C$$

$$-2+1 = -1(0)A+(0)B+1C$$

$$-1 = 0A+(0)B+1C$$

$$-1 = C$$

Put x= 0 in equation 2 , we get

$$2(1)+1 = 1(1+1)A+(1+1)B+(1)^2C$$

$$2(1)+1 = 1(2)A+(2)B+(1)^2C$$

$$3 = 2A+2B+C$$

$$3 = 2A+2(1)+(-1)$$

$$3 = 2A+1$$

$$2A = 2$$

$$A = 1$$

Put the values of A , B , and C equation 1 , we get

$$\frac{2x+1}{x^2(x+1)} = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{(x+1)}$$

3 . Resolve in to partial fractions $\frac{3x+2}{(x+1)(x^2-1)}$

Solution: $\dfrac{3x+2}{(x+1)(x^2-1)} = \dfrac{3x+2}{(x+1)(x-1)(x+1)}$

$$= \frac{3x+2}{(x-1)(x+1)^2}$$

Let $\dfrac{3x+2}{(x-1)(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{c}{(x+1)^2} ----------1$

$$\therefore \frac{3x+2}{(x-1)(x+1)^2} = \frac{(x+1)^2 A + (x+1)(x-1)B + (x-1)C}{(x-1)(x+1)^2}$$

$$\therefore \quad 3x+2 = (x+1)^2 A + (x+1)(x-1)B + (x-1)C ----------2$$

Put x =1 in equation 2 , we get

$$3(1)+2 = (1+1)^2 A + (1+1)(1-1)B + (1-1)C$$

$$5 = (2)^2 A + (2)(0)B + (0)C$$

$$5 = 4A + 0B + (0)C$$

$$5 = 4A$$

$$\therefore \quad A = \frac{5}{4}$$

Put x = -1 in equation 2 , we get

$$3(-1)+2 = (-1+1)^2 A + (-1+1)(-1-1)B + (-1-1)C$$

$$-3+2 = (0)^2 A + (0)(-2)B + (-2)C$$

$$-1 = 0A + 0B + (-2)C$$

$$-1 = -2C$$

$$\therefore C = \frac{-1}{-2} = \frac{1}{2}$$

Put x = 0 in equation 2 , we get

$$3(0)+2 = (0+1)^2 A + (0+1)(0-1)B + (0-1)C$$

$$2 = A + (1)(-1)B + (-1)C$$

$$2 = A - B - C$$

$$2 = \frac{5}{4} - B - \frac{1}{2}$$

$$B = \frac{5}{4} - \frac{1}{2} - 2$$

$$B = \frac{5-2-8}{4}$$

$$\therefore \quad B = \frac{-5}{4}$$

Put the values of A, B and in 1 , we get

$$\frac{3x+2}{(x-1)(x+1)^2} = \frac{\frac{5}{4}}{x-1} + \frac{\frac{-5}{4}}{x+1} + \frac{\frac{1}{2}}{(x+1)^2}$$

EXERCISE:

Resolve in to partial fraction

1. $\dfrac{1}{(x+1)^2\,(x+2)}$ 2. $\dfrac{x^2-2x+7}{(x+1)\,(x-1)^2}$ 3. $\dfrac{2x+3}{x^2(x-1)}$

2. $\dfrac{x^2}{(x+1)\,(x-2)^2}$ 5. $\dfrac{2x-3}{(x+1)\,(x^2-1)}$ 6. $\dfrac{x^2+x+1}{(x-2)\,(x^2-4)}$

7. $\dfrac{3x^2+5x}{(x^2-1)\,(x+1)}$ 8. $\dfrac{x^2}{(x+1)\,(x+2)^2}$ 9. $\dfrac{2x^2+5}{(x-1)^2\,(x-3)}$

CASE III : When denominator contains non repeated irreducible quadratic factor :

$$\frac{ax+b}{(x-\alpha)(x^2+\beta)} = \frac{A}{(x-\alpha)} + \frac{Bx+C}{(x^2+\beta)}$$

Here $(x-\alpha)$ is linear non- repeated factor and $(x^2+\beta)$ is an irreducible quadratic factor

SOLVED EXAMPLES:

1 Resolve in to partial fraction $\dfrac{3x-2}{(x+2)(x^2+4)}$

Solution : Let $\dfrac{3x-2}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$ ----------------1

$$\frac{3x-2}{(x+2)(x^2+4)} = \frac{(x^2+4)A+(Bx+C)(x+2)}{(x+2)(x^2+4)}$$

$$3x-2 = (x^2+4)A+(Bx+C)(x+2) \text{-------} 2$$

Put x = - 2 in equation 2 , we get

$$3(-2)-2 = ((-2)^2+4)A+(B(-2)+C)(-2+2)$$

$$-6-2 = (4+4)A+(-2B+C)(0)$$

$$-8 = (8)A+(0)$$

$$-8 = (8)A$$

$$A = \frac{-8}{8} = -1$$

Put x = 0 in equation 2 , we get

$$3(0)-2 = ((0)^2+4)A+(B(0)+C)((0)+2)$$

$$0-2=(0+4)A+(0)+C)((0)+2)$$
$$-2=(4)A+C(2)$$
$$-2=(4)(-1)+C(2)$$
$$-2=-4+2C$$
$$-2+4=2C$$
$$2=2C$$
$$C=1$$

Put x = 1 in equation 2

$$3(1)-2=(1^2+4)A+(B(1)+C)(1+2)$$
$$3-2=(1+4)A+(B+C)(1+2)$$
$$1=(5)A+(B+C)(3)$$
$$1=5(-1)+3B+3(1)$$
$$1+5-3=3B$$
$$3=3B$$
$$B=1$$

Put the values of A , B , C in equation 1

$$\frac{3x-2}{(x+2)(x^2+4)}=\frac{-1}{x+2}+\frac{1x+1}{x^2+4}$$

2) Resolve into partial fractions $\frac{x^2+23x}{(x+3)(x^2+1)}$

Given fraction

$$\frac{x^2+23x}{(x+3)(x^2+1)}=\frac{A}{(x+3)}+\frac{(Bx+C)}{(x^2+1)} \quad \ldots (1)$$

$$\therefore \quad \frac{x^2+23x}{(x+3)(x^2+1)}=\frac{A(x^2+1)+(Bx+C)(x+3)}{(x+3)(x^2+1)}$$

Comparing numerator's

$$\therefore \quad x^2+23x = A(x^2+1)+(Bx+C)(x+3) \quad \ldots (2)$$

Put x = – 3 in equation (2)

$$(-3)^2+23(-3) = A((-3)^2+1)$$
$$9-69 = A(9+1)$$
$$-60 = A(10)$$
$$\therefore \quad A = -6$$

Put x = 0, A = – 6 in equation (2), we get

$$0^2+23(0) = -6(0^2+1)+(B(0)+C)(0+3)$$
$$0 = -6(1)+(C)(3)$$

$$0 = -6 + 3C$$
$$C = \frac{6}{3}$$
$$C = 2$$

Put x = 1, A = – 6 and C = 2 in equation (1), we get

$$(1)^2 + 23(1) = -6\,((1)^2 + 1) + (B(1) + 2)\,(1 + 3)$$
$$1 + 23 = -6(2) + (B + 2)\,(4)$$
$$24 = -12 + 4B + 8$$
$$24 = -4 + 4B$$
$$24 + 4 = 4B$$
$$28 = 4B$$
$$\therefore \quad B = \frac{28}{4}$$
$$B = 7$$

Putting the value of A, B, C in equation (1)

$$\frac{x^2 + 23x}{(x + 3)\,(x^2 + 23x)} = \frac{-6}{(x + 3)} + \frac{(7x + 2)}{(x^2 + 1)}$$

EXERCISE:

1. $\frac{2x + 1}{(x - 1)\,(x^2 + 1)}$
2. $\frac{x^2 + 36x + 6}{(x - 1)\,(x^2 + 2)}$
3. $\frac{2x - 3}{(x + 1)\,(x^2 + 4)}$
4. $\frac{x^2 - x + 3}{(x - 2)\,(x^2 + 1)}$
5. $\frac{x - 5}{x^3 + x^2 - 5x}$
6. $\frac{x^2 + 1}{x^3 + 1}$
7. $\frac{x - 2}{x^3 + 1}$
8. $\frac{x}{x^3 - 1}$
9. $\frac{3x^2 + 17x + 14}{x^3 - 8}$
10. $\frac{3x - 2}{(x + 2)\,(x^2 + 4)}$
11. $\frac{x^2 + 23x}{(x - 3)\,(x^2 + 1)}$
12. $\frac{x^2 - 2x + 3}{(x^3 + x)}$

Partial fraction of improper fraction

Solved examples:

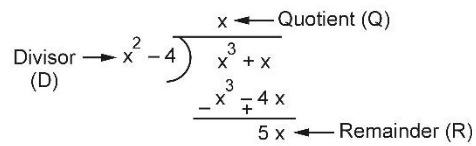
1) Resolve into partial fractions : $\frac{x^3 + x}{x^2 - 4}$

Solution : Given fraction $\frac{x^3 + x}{x^2 - 4}$ is a improper fraction.

We first convert this improper fraction into proper fraction by division method.
Divide numerator by denominator

$$\frac{x^3 + x}{x^2 - 4} = Q + \frac{R}{D}$$

$$\frac{x^3 + x}{x^2 - 4} = x + \underbrace{\frac{5x}{x^2 - 4}}_{\text{Proper fraction}} \quad .... (1)$$



Consider $\quad \frac{5x}{x^2 - 4} = \frac{5x}{(x + 2)(x - 2)}$

Let, $\quad \frac{5x}{(x + 2)(x - 2)} = \frac{A}{(x + 2)} + \frac{B}{(x - 2)} \quad \ldots (2)$

$\therefore \quad \frac{5x}{(x + 2)(x - 2)} = \frac{A(x - 2) + B(x + 2)}{(x + 2)(x - 2)}$

Comparing numerators

$\therefore \quad 5x = A(x - 2) + B(x + 2) \quad \ldots (3)$

To find A, Put x = – 2 in equation (3), we get

$\therefore \quad 5(-2) = A(-2 - 2) + B(0)$

$\therefore \quad -10 = A(-4)$

$\therefore \quad 10 = A(4)$

$\therefore \quad A = \frac{10}{4} \qquad \therefore \quad A = \frac{5}{2}$

To find B, Put x = 2 in equation (3), we get

$5(2) = A(0) + B(2 + 2)$

$\therefore \quad 10 = B(4)$

$\therefore \quad B = \frac{10}{4} \qquad \therefore \quad B = \frac{5}{2}$

Putting the value of A, B in equation (2), we get

$$\frac{5x}{(x^2 - 4)} = \frac{5/2}{(x + 2)} + \frac{5/2}{(x - 2)}$$

Equation (1) becomes

$$\frac{x^3 + x}{x^2 - 4} = x + \frac{5}{2(x + 2)} + \frac{5}{2(x - 2)}$$

2. Resolve into partial fractions : $\frac{x^2 + 1}{x^2 - 1}$

Solution : Given fraction $\frac{x^2 + 1}{x^2 - 1}$ is a improper fraction.

We first convert this improper fraction into proper fraction by division method.

Divide numerator by denominator

$$\frac{x^2+1}{x^2-1} = Q + \frac{R}{D}$$

$$\frac{x^2+1}{x^2-1} = 1 + \underbrace{\frac{2}{x^2-1}}_{\text{Proper fraction}} \quad \ldots (1)$$

Divisor (D) — $x^2 - 1$) $x^2 + 1$ (1 — Quotient (Q)

$\quad - x^2 \mp 1$

$\quad$ 2 — Remainder (R)

Consider, $$\frac{2}{x^2-1} = \frac{2}{(x+1)(x-1)}$$

Let, $$\frac{2}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} \quad \ldots (2)$$

$$\therefore \quad \frac{2}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

Comparing numerators

$$\therefore \quad 2 = A(x-1) + B(x+1) \quad \ldots (3)$$

To find A, Put $x = -1$ in equation (3), we get

$$\therefore \quad 2 = A(-1-1) + B(0)$$

$$\therefore \quad 2 = A(-2)$$

$$\therefore \quad A = \frac{2}{-2} \qquad \therefore \quad A = -1$$

To find B, Put $x = 1$ in equation (3), we get

$$2 = A(0) + B(1+1)$$

$$\therefore \quad 2 = B(2)$$

$$\therefore \quad B = \frac{2}{2}$$

$$\therefore \quad B = 1$$

Putting the value of A, B in equation (2), we get

$$\frac{2}{x^2-1} = \frac{-1}{(x+1)} + \frac{1}{(x-1)}$$

Equation (1) becomes

$$\frac{x^2+1}{x^2-1} = 1 - \frac{1}{(x+1)} + \frac{1}{(x-1)}$$

Exercise:

Resolve into partial fractions:

1. $\dfrac{x^3 + 1}{x^2 + 2x}$
2. $\dfrac{x^4}{x^3 + 1}$
3. $\dfrac{x^3 + 1}{x^2 + 6x}$
4. $\dfrac{x^3 + x}{x^2 - 9}$
5. $\dfrac{x^2 + x}{x^2 - 4}$
6. $\dfrac{x^4}{x^3 - 1}$

Worksheet:

Unit 2
Trigonometry

Course Outcome: Utilize basic concepts of trigonometry to solve elementary engineering problems.

Unit outcome:

a. Apply the concept of compound angle, allied angle, and multiple angles to solve the given simple engineering problem(s).
b. Apply the concept of Sub- multiple angle to solve the given simple engineering related problem(s).
c. Employ concept of factorization and de-factorization formulae to solve the given simple engineering problem(s).
d. Investigate given simple problems utilizing inverse trigonometric ratios.

Introduction: Trigonometry is a study of relationships in Mathematics involving lengths, heights and angles of different triangles. Presently Trigonometry finds wide applications in engineering faculties like Applied Mechanics, Electrical Technology, Basic Electronics, and Computer Engineering etc.

Subtopic: Compound angles, Allied angles, Multiple and Submultiple angles.

Significance : This sub-topic is used to find trigonometric ratios of angles other than standard angles.

Some important formulae:

I. Fundamental Identities:

- $sin^2\theta + cos^2\theta = 1$
- $1 + tan^2\theta = sec^2\theta$
- $1 + cot^2\theta = cosec^2\theta$

II. For a right–angled triangle

- $\sin\theta = \frac{opposite\ side}{hypotenuse}$
- $\sec\theta = \frac{hypotenuse}{adjacent\ side}$
- $\cos\theta = \frac{adjacent\ side}{hypotenuse}$
- $\operatorname{cosec}\theta = \frac{hypotenuse}{opposite\ side}$
- $\tan\theta = \frac{opposite\ side}{adjacent\ side}$
- $\cot\theta = \frac{adjacent\ side}{opposite\ side}$

Also we have,

- $\text{cosec } \theta = \frac{1}{\sin \theta}$
- $\sec \theta = \frac{1}{\cos \theta}$
- $\cot \theta = \frac{1}{\tan \theta}$
- $\tan \theta = \frac{\sin \theta}{\cos \theta}$
- $\cot \theta = \frac{\cos \theta}{\sin \theta}$
- $\text{cosec } \theta \times \sin \theta = 1$
- $\sec \theta \times \cos \theta = 1$
- $\cot \theta \times \tan \theta = 1$

III. Measures of an angle : In practice we use two systems to measure the angle.

a) Sexagesimal system (Degree) : In this system , the unit of measurement is degree.

b) Circular systems (Radian) : In this system , the unit of measurement is radian.

Relation between degrees and radians:

Notation**:** Any angle can be measured in degrees = θ^o and in radians = θ^c

1. Conversion of angle in degrees to angle in radians

$$\theta^c = \theta^o \times \frac{\pi}{180}$$

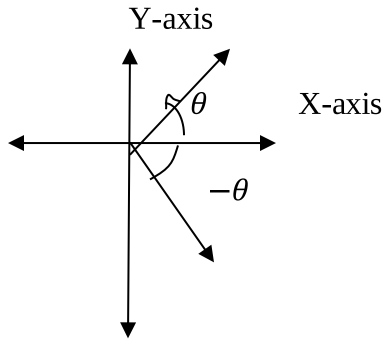
2. Conversion of radians to degree

$$\theta^o = \theta^c \times \frac{180}{\pi}$$

The following table shows the conversion of degree measure to radian measure of standard angles

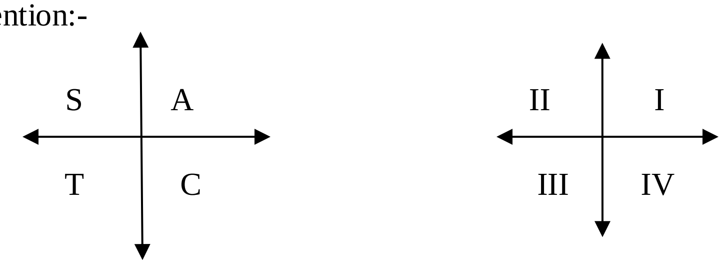
θ^o	θ^c
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
180	π
270	$\frac{3\pi}{2}$
360	2π

IV. **Negative Angle:**



Definition: The angle is said to be negative if it is measured in the clock-wise direction.

V. Sign Convention:-



1. A - All trigonometric ratios are positive
2. S – Sine ratio is positive
3. T – Tangent ratio is positive
4. C – Cosine ratio is positive

Table for values of six trigonometric ratios of 0^0 , 30^0, 45^0, 60^0, 90^0 , 180^0.

θ / Ratios	0^0	$30^0\left(\frac{\pi}{6}\right)$	$45^0\left(\frac{\pi}{4}\right)$	$60^0\left(\frac{\pi}{3}\right)$	$90^0\left(\frac{\pi}{2}\right)$	$180^0(\pi)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1
$\text{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞

Content of the Compound Angle and Allied angle:

Compound angle:

Definition: The angle obtained by addition or subtraction of given angles is called compound angle.

For e.g. A+B, A-B are called compound angles

e. g. $A = 45°, \; B = 30°$ then

$A + B = 45° + 30° = 75°$ and

$A - B = 45° - 30° = 15°$ are compound angles.

Trigonometric ratios of compound angles (Without proof) :

1) $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

2) $\sin(A - B) = \sin A \cdot \cos B - \cos A \sin B$

3) $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

4) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

5) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

6) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

SOLVED EXAMPLES:

Without using calculator, find the value of

1) $\sin 15°$ 2) $\cos 75°$

Solution : 1) $15° = 45° - 30°$

$\therefore \sin 15° = \text{Sin}(45° - 30°)$

$= \sin 45° \cdot \cos 30° - \cos 45° \cdot \sin 30°$

$= \left(\dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2}\right) - \left(\dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}\right)$

$= \dfrac{\sqrt{3}}{2\sqrt{2}} - \dfrac{1}{2\sqrt{2}}$

$\therefore \sin 15° = \mathbf{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}$

2) $75° = 45° + 30°$

$\therefore \cos 75° = \cos(45° + 30°)$

$= \cos 45° \cdot \cos 30° - \sin 45° \cdot \sin 30°$

$= \left(\dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2}\right) - \left(\dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}\right)$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$\therefore \quad \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

3) If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$ find $\tan(A + B)$

Solution : $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}}$$

$$= \frac{\frac{3+2}{6}}{\frac{6-1}{6}} = \frac{5}{5}$$

$$= 1$$

4) If $\tan(x + y) = \frac{3}{4}$ and $\tan(x - y) = \frac{8}{15}$ then show that $\tan 2x = \frac{77}{36}$

Solution :

As $2x = (x + y) + (x - y)$

$\therefore$ L.H.S.$= \tan 2x = \tan[(x + y) + (x - y)]$

$$= \frac{\tan(x + y) + \tan(x - y)}{1 - \tan(x + y) \cdot \tan(x - y)}$$

$$= \frac{\frac{3}{4} + \frac{8}{15}}{1 - \frac{3}{4} \cdot \frac{8}{15}}$$

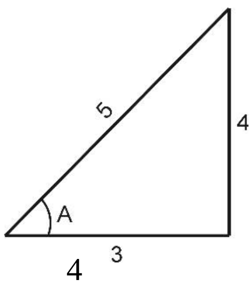
$$= \frac{\frac{3}{4} + \frac{8}{15}}{1 - \frac{24}{60}}$$

$$= \frac{\frac{45+32}{60}}{\frac{60-24}{60}}$$

$$= \frac{77}{36} = \text{R.H.S.}$$

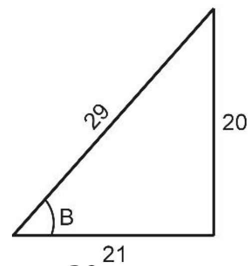
5) If $\cos A = \frac{-3}{5}$, $\sin B = \frac{20}{29}$, where A and B are the angles in the third and second quadrant respectively, find tan (A + B)

Given : $\cos A = \frac{-3}{5}$



$\tan A = \frac{4}{3}$

$\sin B = \frac{20}{29}$



$\tan B = \frac{20}{21}$

As A is in the third quadrant, tan A is positive and B is in the second quadrant, tan B is negative.

$\boxed{\tan A = \frac{4}{3}}$ $\boxed{\tan B = \frac{-20}{21}}$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$= \frac{\frac{4}{3} + \left(\frac{-20}{21}\right)}{1 - \left(\frac{4}{3} \times \frac{-20}{21}\right)} = \frac{\frac{4}{3} - \frac{20}{21}}{1 + \frac{80}{63}}$$

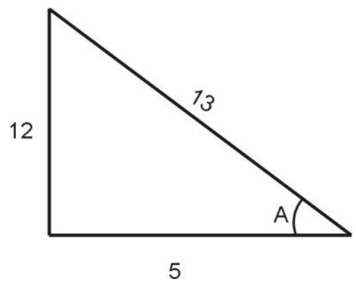
$$= \frac{\frac{84 - 60}{63}}{\frac{63 + 80}{63}}$$

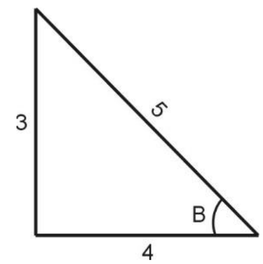
$$\tan(A+B) = \frac{24}{143} \quad \text{OR} \quad 0.168$$

6) If ∠A and ∠B are both obtuse angles and $\sin A = \frac{12}{13}$ and $\cos B = \frac{-4}{5}$, find sin (A + B).

Solution : Given

$\sin A = \frac{12}{13}$ $\cos B = \frac{-4}{5}$





$\cos A = \frac{5}{13}$ | $\sin B = \frac{3}{5}$

$\because$ A and B are obtuse (More than 90º and less than 180º)

$\therefore$ A is the second quadrant, cos A = –5/13

B is the second quadrant, sin B = 3/5

$\boxed{\cos A = \frac{-5}{13}}$ | $\boxed{\sin B = \frac{3}{5}}$

$\therefore \sin (A + B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{12}{13}\right) \times \left(\frac{-4}{5}\right) + \left(\frac{-5}{13}\right)\left(\frac{3}{5}\right)$$

$$= \frac{-48}{65} + \frac{-15}{65}$$

$$= \frac{-48-15}{65} = \frac{-63}{65}$$

EXERCISE:

1. Without using calculator, find the value

   1) cos 105° 2) tan 75° 3) tan 15°

   4) sin 75° 5) tan 105° 6) sin 105°

2. Evaluate without using calculator $\frac{\tan 85° - \tan 40°}{1 + \tan 85° \cdot \tan 40°}$

3. Prove that $\frac{1 - \tan 2\theta \cdot \tan \theta}{1 + \tan 2\theta \cdot \tan \theta} = \frac{\cos 3\theta}{\cos \theta}$

4. If $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{4}$, where $0 < A < \frac{\pi}{2}$, $\pi < B < \frac{3\pi}{2}$, find sin (A + B)

5. If $\sin = \frac{-5}{13}$, $\cos \beta = \frac{-7}{25}$ and α, β lies in the third quadrant, find sin (α - β)

6. If A and B both obtuse angles and $\sin A = \frac{5}{13}$ and $\cos B = \frac{-4}{5}$, then find sin (A + B) and the quadrant of angle A + B.

7. If A and B are obtuse angles such that $\sin A = \frac{5}{13}$ and $\cos B = \frac{-4}{5}$, find tan (A + B).

Allied Angles

If the sum or difference of the measures of two angles is either zero or is an integral multiple of 90°, that is $\pm n \cdot \frac{\pi}{2}$ then these angles are called Allied angles.

For any angle θ, let α be It's allied angle then

$\alpha + \theta = n \cdot \frac{\pi}{2} \Rightarrow \alpha = n \cdot \frac{\pi}{2} - \theta$ allied angle of θ

and $\alpha - \theta = n \cdot \frac{\pi}{2} \Rightarrow \alpha = n \cdot \frac{\pi}{2} + \theta$ allied angle of θ.

For any angle of θ ; $n \cdot \frac{\pi}{2} \pm \theta$ are allied angle of θ.

In general the above results can be written as

If n is an even integer

$\sin\left(n\frac{\pi}{2} \pm \theta\right) \pm \sin\theta$ $\quad$ $\cos\left(n\frac{\pi}{2} \pm \theta\right) = \pm\cos\theta$

If n is an odd integer

$\sin\left(n\frac{\pi}{2} \pm \theta\right) = \pm\cos\theta$ $\quad$ $\cos\left(n\frac{\pi}{2} \pm \theta\right) = \pm\sin\theta$

The algebraic sign is settled down by knowing the quadrant in which the angle $\left(n\frac{\pi}{2} \pm \theta\right)$ lies.

Note that $\sin n\pi = 0$

$\cos n\pi = -1$	If n is odd integer
$\cos n\pi = 1$	If n is even integer

For the angles $\left(\frac{\pi}{2} \pm \theta\right)$ i.e. (90° + θ) and $\left(\frac{3\pi}{2} \pm \theta\right)$ i.e. (270° ± θ)

Make the changes as follows :

sin ↔ cos, tan ↔ cot, sec ↔ cosec and for the angles (π ± θ) i.e. (180° ± θ) and (2π ± θ) i.e. (360° ± θ) there is no change in trigonometric ratios.

i.e. sin ↔ sin, cos ↔ cos etc.

- The algebraic sign is settled down by knowing the quadrant in which the angle lies .

e.g. $\sin\left(\frac{3\pi}{2} - \theta\right) = ?$

For angle $\left(\frac{3\pi}{2} - \theta\right)$ sin changes to cos and angle $\left(\frac{3\pi}{2} - \theta\right)$ lies in IIIrd - quadrant and in IIIrd quadrant sin ratio is negative.

$$\therefore \quad \sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$$

Solved Examples:

1) Without using calculator, find the value of i) sin (– 765°) ii) sec (3660°)

Solution:

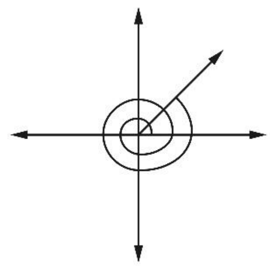
i) sin (– 765°)

$\boxed{\sin(-\theta) = -\sin\theta}$

$= -\sin(765°)$

$= -\sin(8 \times 90° + 45°)$

$= -\sin 45°$

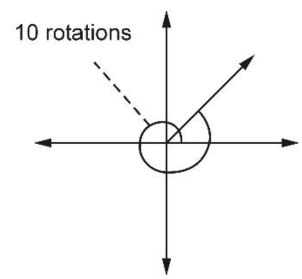


∴ 765° lies in first quadrant,
where sin is +ve
n = 8 (even) function remains same.

ii) sec (3660°)

$= \sec(40 \times 90° + 60°)$

$= +\sec 60° = 2$



3660 lies in I[st] quadrant,
where sec. is positive.
n = 40 (even) function remains same.

2) Evaluate : $\dfrac{\tan 66 + \tan 69}{1 - \tan 66 \cdot \tan 69}$

Solution: $\because \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = \tan(A + B)$

$$\therefore \frac{\tan 66 + \tan 69}{1 - \tan 66 \cdot \tan 69} = \tan(66 + 69)$$

$= \tan(135°)$

$= \tan(90° + 45°)$

$= -\cot 45°$ $\boxed{\tan(90° + \theta) = -\cot\theta}$

$= -1$

3) Without using calculator, find the value of

sin 420° cos 390° + cos (– 300°) sin (– 330°)

Solution : Given $\sin 420° \cos 390° + \cos(-300°)\sin(-330°)$

$$\sin(420°) = \sin(4 \times 90° + 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos(390°) = \cos(360° + 30°)$$

$$= \cos(4 \times 90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\cos(-300°) = \cos(300°)$$

$$= \cos(3 \times 90° + 30°) = \sin 30° = \frac{1}{2}$$

$$\sin(-330°) = -\sin(330°) = -\sin(360° - 30°)$$

$$= -\sin(90° \times 4 - 30°) = -(-\sin 30°)$$

$$= \sin 30° = \frac{1}{2}$$

$\therefore \sin 420° \cos 390° + \cos(-300°)\sin(-330°)$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\ \frac{1}{2}$$

$$= \frac{3}{4} + \frac{1}{4}$$

$$= \frac{4}{4} = 1$$

EXERCISE:

1) Without using calculator, find the value of

1) $\sin 210°$ 2) $\cos 330°$ 3) $\sec^2(-765°)$

4) $\cot(-710°)$ 5) $\mathrm{cosec}(-960°)$

2) Without using calculator, find the value of

1) $\dfrac{\sec^2(135°)}{\cos(-240°) - 2\sin(930°)}$

2) Prove that : $\cos(570°) \cdot \sin(510°) + \sin(-330°) \cdot \cos(-390°) = 0$

3) $\tan(585°) \cdot \cot(-495°) - \cot(405°) \cdot \tan(-495°)$

4) $\sin(150°) - \tan(315°) + \cos(300°) + \sec^2(3660°)$

5) $\cos(-1125°) + \tan(-945°) + \cos(495°)$

6) $\cos(225°)\cos(675°) + \sin(585°)\sin(315°)$

7) $\sin 660° \sin 480° - \cos 120° \sin 330°$

Multiple Angles:

Trigonometric ratios of multiple angle:

The integral multiple of A i.e. 2A, 3A, 4A, ... are called multiple angles of A.

e.g. If A = 60° then 2A = 120°, 3A = 180°, 4A = 240° etc.

Trigonometric ratios of 2θ:

1) $\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$= \dfrac{2\tan\theta}{1+\tan^2\theta}$

2) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$= 2\cos^2\theta - 1$

$= \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

From the above we can deduce

$1 - \cos 2\theta = 2\sin^2\theta$

$1 + \cos 2\theta = 2\cos^2\theta$

3) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Trigonometric ratios of 3θ:

1) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
2) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
3) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

SOLVED EXAMPLES:

1) If sin A = 0.4, find sin 3A

Solution: Given sin A= 0.4

$\sin 3A = 3\sin A - 4\sin^3 A$

$= 3(0.4) - 4(0.4)^3$

$= 1.2 - 0.256$

$= 0.944$

2) If $\cos A = \dfrac{1}{2}$, find the value of cos 3A

Solution: Given $\cos A = \dfrac{1}{2}$

$\cos 3A = 4\cos^3 A - 3\cos A$

$= 4\left(\dfrac{1}{2}\right)^3 - 3\left(\dfrac{1}{2}\right)$

$= 4\left(\dfrac{1}{8}\right) - \dfrac{3}{2}$

$$= \frac{1}{2} - \frac{3}{2}$$

$$= \frac{-2}{2} = -1$$

3) If A = 30º, verify that

i) $\sin 2A = 2 \sin A \cdot \cos A$

ii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

Solution: i) L.H.S. = $\sin 2A$

$$= \sin(2 \times 30^\circ) = \sin 60^\circ$$

$$= \frac{\sqrt{3}}{2}$$

R.H.S. = $2 \sin A \cdot \cos A$

$$= 2 \sin 30^\circ \cdot \cos 30^\circ$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

∴ L.H.S. = R.H.S.

ii) L.H.S. = $\cos 2A$

$$= \cos 2(30^\circ)$$

$$= \cos 60^\circ$$

$$= \frac{1}{2}$$

R.H.S. = $\dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

$$= \frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ}$$

$$= \frac{1 - \left(\frac{1}{\sqrt{3}}\right)^2}{1 + \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{2}$$

∴ L.H.S. =R.H.S.

4) Prove that : $\dfrac{\sin 4\theta + \sin 2\theta}{1 + \cos 2\theta + \cos 4\theta} = \tan 2\theta$

Solution : Consider L.H.S. = $\dfrac{\sin 4\theta + \sin 2\theta}{1 + \cos 2\theta + \cos 4\theta}$

$\sin 4\theta = 2 \sin 2\theta \cdot \cos 2\theta$	$1 + \cos 4\theta = 2 \cos^2 2\theta$

$$\text{L.H.S.} = \frac{2\sin 2\theta \cdot \cos 2\theta + \sin 2\theta}{2\cos^2 2\theta + \cos 2\theta}$$

$$= \frac{\sin 2\theta\,(2\cos 2\theta + 1)}{\cos 2\theta\,(2\cos 2\theta + 1)}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \tan 2\theta$$

$$= \text{R.H.S.}$$

EXERCISE:

1) If $\cos\alpha = 0.4$, find $\cos 3\alpha$
2) If $\sin A = 0.4$, find $\cos 2A$ using multiple angle formula
3) If $\sin A = \frac{1}{2}$, find $\sin 3A$.
4) If $\sin A = \frac{3}{5}$, find the value of $\sin 2A$.
5) If $A = 60°$ verify the result $\sin 3A = 3\sin A - 4\sin^3 A$
6) Prove that $\frac{\sin 2\theta + \cos\theta}{1 - \cos 2\theta + \sin\theta} = \cot\theta$
7) Prove : $\frac{1 + \sin 2A + \cos 2A}{1 + \sin 2A - \cos 2A} = \cot A$

Submultiple Angles:

Let A be the given angle then $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$, ... are called submultiples angles of A.

e.g. If $A = 60°$ then $\frac{A}{2} = 30$, $\frac{A}{3} = 20$, $\frac{A}{4} = 15°$ ...... etc.

Trigonometric ratios of any angle in terms of its sub-multiple angle:

1) $\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$

$= 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$= 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

2) $\sin\theta = 2\sin\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{2}\right)$

3) $\tan\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$

Solved Examples:

1) If $\tan\left(\frac{A}{2}\right) = \frac{1}{\sqrt{3}}$, find the value of $\cos A$.

Solution : We know that

$$\cos A = \frac{1 - \tan^2 A/2}{1 + \tan^2 A/2}$$

$$= \frac{1 - (1/\sqrt{3})^2}{1 + (1/\sqrt{3})^2}$$

$$= \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}}$$

$$= \frac{\frac{3-1}{3}}{\frac{3+1}{3}}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

2) Prove that $\dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$

Solution : L.H.S.= $\dfrac{\sin\theta}{1+\cos\theta}$

$$= \frac{2\sin\theta/2 \cdot \cos\theta/2}{2\cos^2\theta/2}$$

$\sin\theta = 2\sin\theta/2 \cdot \cos\theta/2$
$1 + \cos\theta = 2\cos^2\theta/2$

$$= \frac{2\sin\theta/2 \cdot \cos\theta/2}{2\cos\theta/2 \cdot \cos\theta/2}$$

$$= \frac{\sin\theta/2}{\cos\theta/2}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

EXERCISE:

1) If $\theta = 45°$, find the value of $\cos\left(\frac{\theta}{2}\right)$
2) If $\tan\dfrac{\alpha}{2} = \dfrac{1}{\sqrt{3}}$ find the value of $\sin\alpha$
3) Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\theta/2$
4) Prove that $\dfrac{\cos A}{1-\sin A} = \dfrac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}$

Factorization and De- Factorization formulae

Significance: De-factorization formulae are used to express product of trigonometric functions to sum or difference and Factorization formulae are used to express sum or difference of trigonometric functions to its product.

Content of the sub-topic:

Factorization:

Factorization formulae (Conversion of sum or difference into product)

1) $\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cdot \cos\left(\frac{C - D}{2}\right)$

2) $\sin C - \sin D = 2 \cos\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{C - D}{2}\right)$

3) $\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cdot \cos\left(\frac{C - D}{2}\right)$

4) $\cos C - \cos D = 2 \sin\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{D - C}{2}\right)$

OR

$$\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{C - D}{2}\right)$$

Solved Examples:

1. Express into product form:
   a. $\sin 20 + \sin 40$
   b. $\sin 70 - \sin 50$
   c. $\cos 3\theta + \cos 2\theta$
   d. $\cos 15 - \cos 75$

Solution:

a. $\sin 20 + \sin 40 = 2 \sin\left(\frac{40+20}{2}\right) \cos\left(\frac{40-20}{2}\right) = 2 \sin 30 \cos 10$

b. $\sin 70 - \sin 50 = 2 \cos\left(\frac{70+50}{2}\right) \sin\left(\frac{70-50}{2}\right) = 2 \cos 60 \sin 10$

c. $\cos 3\theta + \cos 2\theta = 2 \cos\left(\frac{3\theta+2\theta}{2}\right) \cos\left(\frac{3\theta-2\theta}{2}\right) = 2 \cos\left(\frac{5\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)$

d. $\cos 15 - \cos 75 = 2 \sin\left(\frac{15+75}{2}\right) \sin\left(\frac{75-15}{2}\right) = 2 \sin 45 \sin 30$

2. Prove that : $\frac{\sin 3A - \sin A}{\cos 3A + \cos A} = \tan A$

L.H.S. $= \frac{\sin 3A - \sin A}{\cos 3A + \cos A}$

$$\boxed{\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)}$$

$$= \frac{2\cos\left(\frac{3A+A}{2}\right)\cdot\sin\left(\frac{3A-A}{2}\right)}{2\cos\left(\frac{3A+A}{2}\right)\cdot\cos\left(\frac{3A-A}{2}\right)}$$

$$\boxed{\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)}$$

$$= \frac{2\cos\left(\frac{4A}{2}\right)\cdot\sin\left(\frac{2A}{2}\right)}{2\cos\left(\frac{4A}{2}\right)\cos\left(\frac{2A}{2}\right)}$$

$$= \frac{2\cos 2A\cdot\sin A}{2\cos 2A\cdot\cos A}$$

$$= \frac{\sin A}{\cos A}$$

$$= \tan A$$

= R.H.S.

3. Prove that $\dfrac{\sin 4A + \sin 5A + \sin 6A}{\cos 4A + \cos 5A + \cos 6A} = \tan 5A$

Solution : L.H.S. = $\dfrac{\sin 4A + \sin 5A + \sin 6A}{\cos 4A + \cos 5A + \cos 6A}$

Rearranging the terms

$$= \frac{\sin 4A + \sin 6A + \sin 5A}{\cos 4A + \cos 6A + \cos 5A}$$

$$= \frac{2\sin\left(\frac{4A+6A}{2}\right)\cdot\cos\left(\frac{4A-6A}{2}\right) + \sin 5A}{2\cos\left(\frac{4A+6A}{2}\right)\cos\left(\frac{4A-6A}{2}\right) + \cos 5A}$$

$$= \frac{2\sin\left(\frac{10A}{2}\right)\cos\left(\frac{-2A}{2}\right) + \sin 5A}{2\cos\left(\frac{10A}{2}\right)\cos\left(\frac{-2A}{2}\right) + \cos 5A}$$

$$= \frac{2\sin 5A\cdot\cos(-A) + \sin 5A}{2\cos 5A\ \cos(-A) + \cos 5A}$$

$$= \frac{\sin 5A\,(\cancel{2\cos A + 1})}{\cos 5A\,(\cancel{2\cos A + 1})}$$

$$= \frac{\sin 5A}{\cos 5A}$$

$= \tan 5A$

$=$ R.H.S.

4. Prove that $\dfrac{\cos 2A + 2\cos 4A + \cos 6A}{\cos A + 2\cos 3A + \cos 5A} = \cos A - \tan 3A \cdot \sin A$

Solution : L.H.S. $= \dfrac{\cos 2A + 2\cos 4A + \cos 6A}{\cos A + 2\cos 3A + \cos 5A}$

Rearranging the terms

$$= \frac{(\cos 2A + \cos 6A) + 2\cos 4A}{(\cos A + \cos 5A) + 2\cos 3A}$$

$$\boxed{\cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)}$$

$$= \frac{2\cos\left(\frac{2A + 6A}{2}\right)\cos\left(\frac{2A - 6A}{2}\right) + 2\cos 4A}{2\cos\left(\frac{A + 5A}{2}\right)\cos\left(\frac{A - 5A}{2}\right) + 2\cos 3A}$$

$$= \frac{2\cos\left(\frac{8A}{2}\right)\cos\left(\frac{-4A}{2}\right) + 2\cos 4A}{2\cos\left(\frac{6A}{2}\right)\cos\left(\frac{-4A}{2}\right) + 2\cos 3A}$$

$$= \frac{2\cos 4A \cdot \cos(-2A) + 2\cos 4A}{2\cos 3A\ \cos(-2A) + 2\cos 3A}$$

Taking 2 cos 4A common from numerator and 2 cos 3A common from denominator.

$$= \frac{\cancel{2}\cos 4A\ \cancel{(\cos 2A + 1)}}{\cancel{2}\cos 3A\ \cancel{(\cos 2A + 1)}} \qquad \boxed{\cos(-\theta) = \cos\theta}$$

$$= \frac{\cos 4A}{\cos 3A}$$

Put $4A = 3A + A$

$$= \frac{\cos(3A + A)}{\cos 3A}$$

$$\boxed{\cos(A + B) = \cos A \cdot \cos B - \sin A \sin B}$$

$$= \frac{\cos 3A \cdot \cos A - \sin 3A \cdot \sin A}{\cos 3A}$$

$$= \frac{\cancel{\cos 3A}\ \cos A}{\cancel{\cos 3A}} - \frac{\sin 3A\ \sin A}{\cos 3A}$$

$= \cos A - \tan 3A \cdot \sin A$ = R.H.S. $\qquad \boxed{\dfrac{\sin\theta}{\cos\theta} = \tan\theta}$

EXERCISE:

1) Express into product form and evaluate.

1) $\sin 100° + \sin 50°$ 2) $\sin 80° - \cos 70°$ 3) $\sin 80° - \sin 50°$

2) Prove that : $\dfrac{\sin 8\theta + \sin 2\theta}{\cos 8\theta + \cos 2\theta} = \tan 5\theta$

3) Prove that $\dfrac{\sin 7x + \sin x}{\cos 5x - \cos 3x} = \sin 2x - \cos 2x \cot x$

4) Prove that $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

5) Prove that $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \cos 2A - \cot 5A \sin 2A$

6) Prove that $\dfrac{\sin 5A + 2\sin 7A + \sin 9A}{\cos 3A + 2\cos 5A + \cos 7A} = \sin 2A + \cos 2A \tan 5A$

7) Prove that $\dfrac{\sin A + 2\sin 2A + \sin 3A}{\cos A + 2\cos 2A + \cos 3A} = \tan (2A)$

Defactorization formula

(Conversion of product into sum or difference)

1) $2\sin A \cdot \cos B = \sin (A + B) + \sin (A - B)$
2) $2\cos A \cdot \sin B = \sin (A + B) - \sin (A - B)$
3) $2\cos A \cdot \cos B = \cos (A + B) + \cos (A - B)$
4) $2\sin A \cdot \sin B = \cos (A - B) - \cos (A + B)$

Solved Examples:

1. Express as sum or difference of trigonometric functions : $2\cos 75° \cos 15°$

Solution:

Given $2\cos 75° \cos 15°$

$$\begin{aligned} 2\cos A \cdot \cos B &= \cos (A + B) + \cos (A - B) \\ &= \cos (75° + 15°) + \cos (75° - 15°) \\ &= \cos (90°) + \cos (60°) \\ &= 0 + \frac{1}{2} = \frac{1}{2} \end{aligned}$$

2. Prove that $\cos 20° \cdot \cos 40° \cdot \cos 60° \cdot \cos 80° = \dfrac{1}{16}$

Solution : L.H.S.= $\cos 20° \cdot \cos 40° \cdot \cos 60° \cdot \cos 80°$

$$= \cos 20° \cdot \cos 40° \cdot \frac{1}{2} \cdot \cos 80°$$

Multiply and divide by 2

$= \frac{1}{2} \cdot \frac{1}{2} \{2 \cos 20° \cdot \cos 40°\} \cos 80°$

$\boxed{2 \cos A \cdot \cos B = \cos (A + B) + \cos (A - B)}$

$= \frac{1}{4} \{\cos (20° + 40°) + \cos (20° - 40°)\} \cos 80°$

$= \frac{1}{4} \{\cos (60°) + \cos (-20°)\} \cos 80°$ $\boxed{\cos 60° = \frac{1}{2}}$

$= \frac{1}{4} \left\{\frac{1}{2} + \cos 20°\right\} \cos 80°$

$= \frac{1}{4} \left\{\frac{1}{2} \cos 80° + \cos 20° \cdot \cos 80°\right\}$

Multiply and divide by 2

$= \frac{1}{4} \cdot \frac{1}{2} \left\{2 \cdot \frac{1}{2} \cos 80° + 2 \cos 20° \cdot \cos 80°\right\}$

$\boxed{2 \cos A \cdot \cos B = \cos (A + B) + \cos (A - B)}$

$= \frac{1}{8} \{\cos 80° + \cos (20° + 80°) + \cos (20° - 80°)\}$

$= \frac{1}{8} \{\cos 80° + \cos 100° + \cos (-60°)\}$ $\boxed{\cos (-\theta) = \cos \theta}$

$= \frac{1}{8} \{\cos 80° + \cos (180° - 80°) + \cos 60°\}$ $\boxed{\cos (180° - \theta) = -\cos \theta}$

$= \frac{1}{8} \{\cos 80° - \cos 80° + \cos 60°\}$

$= \frac{1}{8} \{\cos 60°\}$ $\boxed{\cos 60° = \frac{1}{2}}$

$= \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16} =$ R.H.S.

3. Prove that $\sin 20° \cdot \sin 40° \cdot \sin 60° \cdot \sin 80° = \frac{3}{16}$

Solution : L.H.S. $= \sin 20° \cdot \sin 40° \cdot \sin 60° \cdot \sin 80°$

$= \sin 20° \cdot \sin 40° \cdot \frac{\sqrt{3}}{2} \cdot \sin 80°$

Multiply and divide by 2

$= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \{2 \sin 20° \cdot \sin 40°\} \sin 80°$

$\boxed{2 \sin A \cdot \sin B = \cos (A - B) - \cos (A + B)}$

$= \frac{\sqrt{3}}{4} \{\cos (20° - 40°) - \cos (20° + 40°\} \sin 80°$

$$= \frac{\sqrt{3}}{4}\{\cos(-20) - \cos 60°\} \sin 80°$$

$$= \frac{\sqrt{3}}{4}\left\{\cos 20° - \frac{1}{2}\right\} \sin 80° \qquad \boxed{\cos(-\theta) = \cos\theta}$$

$$= \frac{\sqrt{3}}{4}\left\{\cos 20° \cdot \sin 80° - \frac{1}{2}\sin 80°\right\}$$

Multiply and divide by 2

$$= \frac{\sqrt{3}}{4} \cdot \frac{1}{2}\left\{2\cos 20° \cdot \sin 80° - 2 \cdot \frac{1}{2}\sin 80°\right\}$$

$$\boxed{2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{\sqrt{3}}{8}\{\sin(20° + 80°) - \sin(20° - 80°) - \sin 80°\}$$

$$= \frac{\sqrt{3}}{8}\{\sin 100° - \sin(-60°) - \sin 80°\}$$

$$= \frac{\sqrt{3}}{8}\{\sin(180° - 80°) + \sin 60° - \sin 80°\} \boxed{\sin(-\theta) = -\sin\theta}$$

$$= \frac{\sqrt{3}}{8}\{\sin 80° + \sin 60° - \sin 80°\}$$

$$= \frac{\sqrt{3}}{8}\{\sin 60°\}$$

$$= \frac{\sqrt{3}}{8}\,\frac{\sqrt{3}}{2} = \frac{3}{16} = \text{R.H.S.}$$

EXERCISE:

1) Express the following as sum or difference of trigonometric functions :
   a) $2\sin 70° \cos 30°$ b) $2\sin 60° \cos 20°$ c) $\sin 11A \,.\, \sin A$
2) Prove that $8\sin 20 \cdot \sin 40 \cdot \cos 10 = \sqrt{3}$
3) Prove that $\cos 10° \cdot \cos 50° \cdot \cos 70° = \frac{\sqrt{3}}{8}$
4) Prove that $\cos 15° \cdot \cos 30° \cdot \cos 60° \cdot \cos 75° = \frac{\sqrt{3}}{16}$
5) Prove that $\sin 10° \cdot \sin 30° \cdot \sin 50° \cdot \sin 70° = \frac{1}{16}$

Inverse Trigonometric Ratios.

Significance: Inverse Trigonometric Ratios are used to find the angle of a given trigonometric function.

Definition

If sin θ = x then θ is called sine inverse of x and is denoted by $\theta = \sin^{-1}x$

Thus, if sin θ = x then $\theta = \sin^{-1} x$

Similarly If $\cos\theta = x$ then $\theta = \cos^{-1}x$

If $\tan\theta = x$ then $\theta = \tan^{-1}x$

If $\cot\theta = x$ then $\theta = \cot^{-1}x$

If $\sec\theta = x$ then $\theta = \sec^{-1}x$

If $\text{cosec}\,\theta = x$ then $\theta = \text{cosec}^{-1}x$

The functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$ and $\sec^{-1}x$ and $\text{cosec}^{-1}x$ are called inverse trigonometric functions.

Note that $\sin^{-1}x \neq \dfrac{1}{\sin x}$

Principal Value of Inverse Trigonometric Ratios:

Consider

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

$$\sin 135° = \frac{1}{\sqrt{2}} \qquad \therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 135°$$

$$\sin 405° = \frac{1}{\sqrt{2}} \qquad \therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 405°$$

Thus we have

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°, 135°, 405°, \ldots\ldots..$$

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is not unique valued function, but it is multivalued function.

From application point of view the smallest of these values is important.This value is called the principal value.

Thus the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$ or $\frac{\pi°}{4}$

Function	Range of principal value
$\sin^{-1} x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[0, \pi]$
$\tan^{-1} x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

<u>Properties of inverse trigonometric function:</u>

Property I:

1) $\sin(\sin^{-1} x) = x$
2) $\cos(\cos^{-1} x) = x$
3) $\tan(\tan^{-1} x) = x$
4) $\cot(\cot^{-1} x) = x$
5) $\sec(\sec^{-1} x) = x$
6) $\text{cosec}(\text{cosec}^{-1} x) = x$

1) $\sin^{-1}(\sin\theta) = \theta$
2) $\cos^{-1}(\cos\theta) = \theta$
3) $\tan^{-1}(\tan\theta) = \theta$
4) $\cot^{-1}(\cot\theta) = \theta$;
5) $\sec^{-1}(\sec\theta) = \theta$;
6) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$

Property II:

1) $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$
2) $\text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$
3) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right)$
4) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$
5) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$
6) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

Property III:

1) $\sin^{-1}(-x) = -\sin^{-1}x$
2) $\cos^{-1}(-x) = \pi - \cos^{-1}x$
3) $\tan^{-1}(-x) = -\tan^{-1}x$
4) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$
5) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$
6) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$

Property IV:

1) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
2) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
3) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

Property V:

1) If $x > 0$, $y > 0$ and $xy < 1$ then

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

2) If x > 0, y > 0 and xy > 1 then

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi$

3) If x > 0, y > 0 then

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

SOLVED EXAMPLES:

Find the principal value of

1) $\sin^{-1}\left(\frac{-1}{2}\right)$ 2) $\cos\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{2}\right)\right]$

Solution:

1)

Let $\sin^{-1}\left(\frac{-1}{2}\right) = \theta$ ∴ $\boxed{\sin^{-1}x = \theta \text{ then } x = \sin\theta}$

∴ $\frac{-1}{2} = \sin\theta$

∴ $-\sin\theta = \frac{1}{2}$ $\boxed{\cos(90° + \theta) = -\sin\theta}$

∴ $\cos(90° + \theta) = \cos 60°$ $\boxed{\cos 60° = \frac{1}{2}}$

$\Rightarrow 90 + \theta = 60$

∴ $\theta = -30°$

Principal value of $\sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{\mathbf{6}}$

2) $\cos\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$= \cos\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$

$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

3) Prove that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution : L.H.S. $= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3+2}{6}}{\frac{6-1}{6}}\right)$$

$$= \tan^{-1}\left(\frac{5/6}{5/6}\right)$$

$$= \tan^{-1}(1)$$

$$= 45° = \frac{\pi}{4}$$

= R.H.S.

4) Prove that $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \cot^{-1}\left(\frac{9}{2}\right)$

Solution : L.H.S. $= \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right)$

We know that If $x > 0$, $y > 0$ and $xy < 1$ then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Here $x = \frac{1}{7} > 0$, $y = \frac{1}{13} > 0$, $xy = \frac{1}{91} < 1$

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\cdot\frac{1}{13}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{13+7}{91}}{\frac{91-1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{20}{90}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

$\tan^{-1}(x) = \cot^{-1}\left(\frac{1}{x}\right)$

$= \cot^{-1}\left(\frac{9}{2}\right)$

$=$ R.H.S.

5) Prove that $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

Solution : L.H.S. $= \tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$

$= \{\tan^{-1}(1) + \tan^{-1}(2)\} + \tan^{-1}(3)$

We know that $x > 0$, $y > 0$ and $xy > 1$ then
$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right) + \pi$

Here $x = 1 > 0$, $y = 2 > 0$, $xy = 2 > 1$

$= \{\tan^{-1}(1) + \tan^{-1}(2)\} + \tan^{-1}(3)$

$= \tan^{-1}\left(\frac{1 + 2}{1 - (1)(2)}\right) + \pi + \tan^{-1}(3)$

$= \tan^{-1}\left(\frac{1 + 2}{1 - 2}\right) + \pi + \tan^{-1}(3)$

$= \tan^{-1}\left(\frac{3}{-1}\right) + \pi + \tan^{-1}(3)$

$= \tan^{-1}(-3) + \pi + \tan^{-1}(3)$ $\quad$ $\tan^{-1}(-x) = -\tan^{-1}x$

$= -\tan^{-1}(3) + \pi + \tan^{-1}(3)$

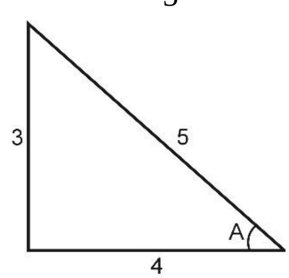
$= \pi =$ R.H.S.

6) Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Solution :

Let $\cos^{-1}\left(\frac{4}{5}\right) = A$

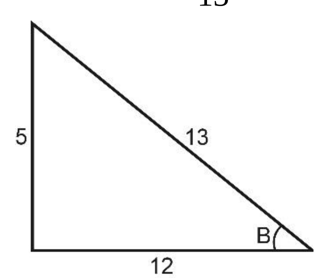
$\cos A = \frac{4}{5}$



$\sin A = \frac{3}{5}$

$\cos^{-1}\left(\frac{12}{13}\right) = B$

$\therefore \cos B = \frac{12}{13}$



$\sin B = \frac{5}{13}$

$\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{48 - 15}{65}$

$$\cos(A + B) = \left(\frac{33}{65}\right)$$

$$A + B = \cos^{-1}\left(\frac{33}{65}\right)$$

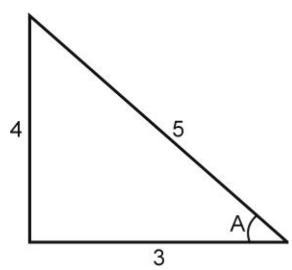
$$\therefore \quad \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

7) Prove that $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{84}{85}\right)$

Solution :

Let $\sin^{-1}\left(\frac{4}{5}\right) = A$

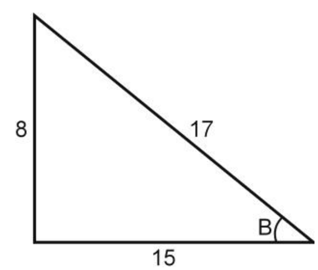
$\therefore \quad \sin A = \frac{4}{5}$



$\cos A = \frac{3}{5}$

$\sin^{-1}\left(\frac{8}{17}\right) = B$

$\therefore \quad \sin B = \frac{8}{17}$



$\cos B = \frac{15}{17}$

Now, $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$$= \frac{4}{5} \cdot \frac{15}{17} + \frac{3}{5} \cdot \frac{8}{17}$$

$$= \frac{60}{85} + \frac{24}{85} = \frac{60 + 24}{85}$$

$$\sin(A + B) = \frac{84}{85}$$

$$A + B = \sin^{-1}\left(\frac{84}{85}\right)$$

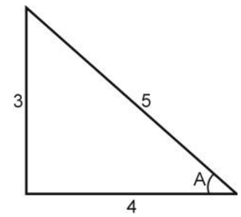
$$\therefore \quad \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{84}{85}\right)$$

8) Prove that $\cos^{-1}\left(\frac{4}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{63}{65}\right)$

Solution:

Let $\cos^{-1}\left(\frac{4}{5}\right) = A$

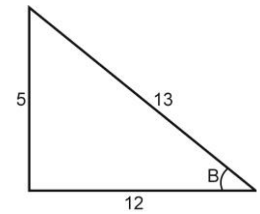
$\therefore \quad \cos A = \frac{4}{5}$



$\sin A = \frac{3}{5}$

$\sin^{-1}\left(\frac{5}{13}\right) = B$

$\therefore \quad \sin B = \frac{5}{13}$



$\cos B = \frac{12}{13}$

Now, $\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

$$= \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13}$$

$$= \frac{48}{65} + \frac{15}{65} = \frac{48 + 15}{65}$$

$$\cos(A - B) = \frac{63}{65}$$

$$A - B = \cos^{-1}\left(\frac{63}{65}\right)$$

$$\therefore \quad \cos^{-1}\left(\frac{4}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{63}{65}\right)$$

Exercise:

1) Find the principal value of:

1) $\sin\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$

2) $\sin\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{2}\right)\right]$

3) $\cos^{-1}\left(\frac{-1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$

4) $\sin^{-1}\left(\frac{3}{5}\right) + \quad \cos^{-1}\left(\frac{3}{5}\right)$

5) $\sec\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

6) $\sin\left[\cos^{-1}\left(\frac{5}{13}\right)\right]$

2) Show that $\tan^{-1}\left(\frac{1}{11}\right) + \tan^{-1}\left(\frac{5}{6}\right) = \frac{\pi}{4}$

3) Show that $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

4) Prove that $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \cot^{-1}(2)$

5) Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{27}{11}\right)$

6) Show that $\cos^{-1}\left(\frac{4}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{63}{65}\right)$

7) Prove that $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{56}{65}\right)$

8) Prove that $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

9) Prove that $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

10) Prove that $\cos^{-1}\left(\frac{20}{29}\right) - \cos^{-1}\left(\frac{4}{5}\right) = \sin^{-1}\left(\frac{24}{145}\right)$

Worksheet:

Unit 3
Co-ordinate Geometry

Course Outcome: Solve basic engineering problems under given conditions of straight lines.

Unit outcome:

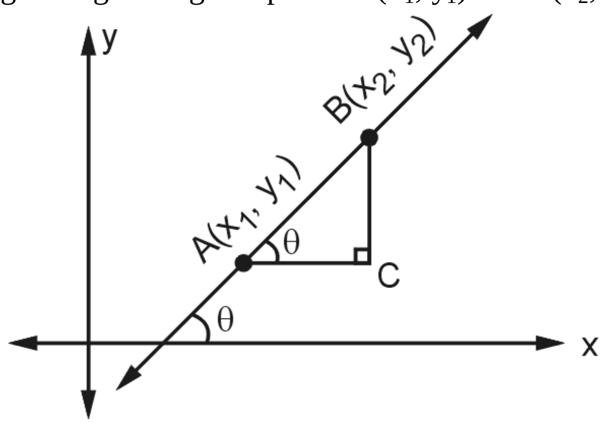
a. Calculate angle between given two straight lines.
b. Formulate equation of straight lines related to given engineering problems.
c. Identify perpendicular distance from the given point to the line.
d. Calculate perpendicular distance between the given two parallel lines.

Introduction: The straight line is one of the most widely used concepts in engineering field. The equation of Straight line and the different concepts related to it i.e. slope, angle between lines are very much essential in understanding the relationship between two variables.

Straight Line- First degree equation in x and y represents a straight line.
Slope of a line: It is tangent ratio of an angle made by the line with positive X-axis.
i.e If θ is the angle made by the line with positive X-axis then $\tan\theta$ is slope of line.
Slope is also called the gradient of the line and it is denoted by m.
$m = \tan\theta$

For example: If $\theta = 30°$, Slope (m) = $\tan 30° = \frac{1}{\sqrt{3}}$

Slope of a Line Passing through Two Points:
If the given line is passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ then from Fig.



Considering ΔABC, we get

$$\tan\theta = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore \quad m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\text{Slope} = m = \frac{\text{difference in y co-ordinate}}{\text{difference in x co-ordinate}}$$

Standard Forms of Equations of Straight Lines:

1) **Slope-point form** : Equation of line having slope 'm' and passing through a point A (x_1, y_1) is $y - y_1 = m(x - x_1)$
2) **Slope-intercept form :** The equation of a line with slope m and whose y intercept 'c' is $y = mx + c$
3) **Two point -intercept form (Double intercept form):** The equation of a line making intercepts a and b on the x-axis and y-axis respectively is
$$\frac{x}{a} + \frac{y}{b} = 1$$
4) **Two point form:** The equation of line passing through the point $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$
5) **General form :** The general equation of straight line is $Ax + By + C = 0$ where A, B and C are real number such that A and B are not simultaneously zero.
we can express $Ax + By + C = 0$ in slope- intercept form as $y = -\frac{A}{B}x - \frac{C}{B}$

Slope $= \frac{-A}{B}$, y-intercept $= \frac{-C}{B}$ and x-intercept $= \frac{-C}{A}$

SOLVED EXAMPLES:

1) Find the slope of a line through points (– 1, – 2) and (– 3, 8).

Solution : Let the given points $A(-1, -2) = A(x_1, y_1)$ and $B(-3, 8) = B(x_2, y_2)$

$$\therefore \quad \text{Slope of AB} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{8 - (-2)}{-3 - (-1)}$$
$$= \frac{10}{-2}$$
$$\text{Slope} = m = -5$$

2) Find the slope of line whose inclination is 60°.

Solution : Given : $\theta = 60°$

We know that Slope of line whose inclination is θ is
$$m = \tan\theta$$
$$\therefore \quad m = \tan 60°$$
$$\therefore \quad m = \sqrt{3}$$

3) Find the equation of line passing through (3, – 4) and having slope $\frac{3}{2}$.

Solution : Given Slope $= (m) = \frac{3}{2}$

and $(x_1, y_1) = (3, -4)$

By using equation of line in slope-point form.

$$y - y_1 = m(x - x_1)$$

$$\therefore \quad y - (-4) = \frac{3}{2}(x - 3)$$

$$\therefore \quad 2(y + 4) = 3(x - 3)$$

$$\therefore \quad 2y + 8 = 3x - 9$$

$$\therefore \quad 3x - 2y - 9 - 8 = 0$$

$$\therefore \quad 3x - 2y - 17 = 0$$

4) Find the equation of straight line passes through the points (3, 5) and (4, 6).

Solution : Equation of line is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\frac{y - 5}{5 - 6} = \frac{x - 3}{3 - 4}$$

$$\frac{y - 5}{-1} = \frac{x - 3}{-1}$$

$$x - y + 2 = 0$$

5) Find the equation of the line whose x-intercept is 3 and y intercept is 4.

Solution : Givenx-intercept = a = 3

y-intercept = b = 4

By using equation of line in two intercept form.

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1$$

$$\therefore \quad \frac{x}{3} + \frac{y}{4} = 1$$

$$\therefore \quad \frac{4x + 3y}{12} = 1$$

$$\therefore \quad 4x + 3y = 12$$

$$\therefore \quad 4x + 3y - 12 = 0$$

6) Find slope and intercepts of the line $\frac{x}{4} - \frac{y}{3} = 2$

Solution: Given equation of line is

$$\frac{x}{4} - \frac{y}{3} = 2$$

$$\frac{3x - 4y}{12} = 2$$

$$3x - 4y = 24$$

$$3x - 4y - 24 = 0$$

$$\text{Slope} = m = \frac{-A}{B} = \frac{-3}{-4} = \frac{3}{4}$$

$$\text{x-intercept} \quad = \frac{-C}{A} = \frac{-(-24)}{3} = \frac{24}{3} = 8$$

$$\text{y-intercept} = \frac{-C}{B} = \frac{-(-24)}{-4}$$

$$= \frac{-24}{4} = -6$$

Exercise:

1) Find the slopes of line passing through points

   1) (3, 4) and (7, 10)
   2) (3, 4) and (– 4, 6)

2) Find the slope of a line whose inclination is

   1) 90° 2) 60°

3) Find slope and intercepts of a line

   1) $5x - 4y + 7 = 0$. 2) $\frac{x}{2} - \frac{y}{3} = \frac{1}{4}$ 3) $3x - 4y + 5 = 0$

4) Find the equation of a line

   1) Passing through (1,7) and having slope 2 units.
   2) With slope $\frac{-3}{2}$ and passing through the point (1, 2).
   3) Passing through (2, 5) and slope $\frac{-4}{5}$.
   4) Passing through (4, –5) having slope –2/3.
   5) Passing through (5, 6) and slope $\frac{-1}{2}$.

5) Find the equation of a straight line passing through points.

   a) (3, 4) and (5, 6)
   b) (1, – 1) and (3, 5)
   c) (4, 3) and (3, – 5)
   d) (7, 4) and (5, 8)

Angle between Two Straight Lines:

If m_1 and m_2 are slopes of the two lines then the angle between two lines is

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

Deduction:

1. If θ= 0 then lines are parallel

$$0 = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$\tan 0 = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$0 = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$m_1 = m_2$$

2 If θ= 90 then lines are perpendicular

$$90 = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$\tan 90 = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$\infty = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$\therefore 1 + m_1 \times m_2 = 0$$

$$m_1 \times m_2 = -1$$

Condition for parallel and perpendicular Lines:

1) Two lines are parallel if their slopes are equal .i. e. $m_1 = m_2$ and converse is also true.
2) Condition for two lines to be perpendicular : Two lines are perpendicular if their product of slopes is – 1. i.e. $m_1 \cdot m_2 = -1$ and converse is also true

Solved Examples:

1) Find the acute angle between the lines $3x - 2y + 4 = 0$ and $2x - 3y - 7 = 0$.

Solution : Given equation of lines

L_1 : $3x - 2y + 4 = 0$

$$\therefore \quad \text{Slope } m_1 = \frac{-3}{-2} = \frac{3}{2}$$

L_2 : $2x - 3y - 7 = 0$

$$\therefore \quad \text{Slope } m_2 = \frac{-2}{-3} = \frac{2}{3}$$

Let 'θ' be the acute angle between the lines

Then $$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$= \left|\frac{\frac{3}{2} - \frac{2}{3}}{1 + \left(\frac{3}{2}\ \frac{2}{3}\right)}\right| = \left|\frac{\frac{(9-4)}{6}}{1+1}\right|$$

$$= \left|\frac{\frac{5}{6}}{2}\right| = \left|\frac{5}{12}\right|$$

$$\tan\theta = \frac{5}{12}$$

$$\theta = \tan^{-1}\left(\frac{5}{12}\right)$$

2) Find the acute angle between the lines 3x – y = 4 and 2x + y = 3.

Solution : Given equation of lines

$$L_1 : \quad 3x - y = 4$$

$$L_1 : \quad 3x - y - 4 = 0$$

$$\therefore \quad \text{Slope } m_1 = \frac{-3}{-1} = 3$$

$$L_2 : \quad 2x + y - 3 = 0$$

$$\therefore \quad \text{Slope } m_2 = \frac{-2}{1} = -2$$

The acute angle between the line is

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$= \tan^{-1}\left|\frac{3 - (-2)}{1 + (3)(-2)}\right|$$

$$= \tan^{-1}\left|\frac{3 + 2}{1 - 6}\right|$$

$$= \tan^{-1}\left|\frac{5}{-5}\right|$$

$$= \tan^{-1}|-1|$$

$$= \tan^{-1}(1)$$

$$\theta = 45° \quad \text{OR} \quad \theta = \frac{\pi}{4}$$

3) Find the angle between the lines 3x – 4y = 420 and 4x + 3y = 420

Solution : Given equations of lines

$$L_1 : \quad 3x - 4y - 420 = 0$$

$$\therefore \quad \text{Slope } m_1 = \frac{-3}{-4} = \frac{3}{4}$$

$$L_2 : \quad 4x + 3y - 420 = 0$$

$$\therefore \quad \text{Slope } m_2 = \frac{-4}{3}$$

The acute angle between the two lines is

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$= \tan^{-1}\left|\frac{\frac{3}{4}-\left(\frac{-4}{3}\right)}{1+\left(\frac{3}{4}\right)\left(\frac{-4}{3}\right)}\right|$$

$$= \tan^{-1}\left|\frac{\frac{3}{4}+\frac{4}{3}}{1+(-1)}\right|$$

$$= \tan^{-1}\left|\frac{\frac{(9+16)}{12}}{1-1}\right|$$

$$= \tan^{-1}\left|\frac{\frac{25}{12}}{0}\right|$$

$$= \tan^{-1}(\infty)$$

$$\theta = 90^\circ \quad \text{OR} \quad \theta = \frac{\pi}{2}$$

The given are perpendicular to each other.

4) Show that the lines $2x + 3y - 5 = 0$ and $4x + 6y - 1 = 0$ are parallel.

Solution : Let $L_1 : 2x + 3y - 5 = 0$

$\therefore$ Slope of L_1 is $m_1 = \frac{-2}{3}$

And $L_2 : 4x + 6y - 1 = 0$

$\therefore$ Slope of L_2 is $m_2 = \frac{-4}{6} = \frac{-2}{3}$

$\therefore m_1 = m_2$

$\therefore$ Given lines are parallel.

5) Prove that lines $3x + 4y + 7 = 0$ and $28x - 21y + 50 = 0$ are perpendicular to each other.

Solution : Let $L_1 : 3x + 4y + 7 = 0$

$\therefore$ Slope of L_1 is $m_1 = \frac{-3}{4}$

and $L_2 : 28x - 21y + 50 = 0$

$\therefore$ Slope of L_2 is $m_2 = \frac{-28}{-21} = \frac{4}{3}$

$m_1 \cdot m_2 = \frac{-3}{4} \cdot \frac{4}{3} = -1$

$\therefore$ Given lines are perpendicular.

6) Find the equation of line passing through $(2, -3)$ and parallel to the line $4x - y + 7 = 0$.

Solution : Given equation of line is

$L_1 :\ 4x - y + 7 = 0$

$$\text{Slope} = m_1 = \frac{-\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-4}{-1}$$

$\therefore \quad m_1 = 4$

Slope of required line is, m_2,

The required line is parallel to given line (L_1)

$\therefore \quad m_2 = 4$

Also required line passes through (2, – 3)

By slope-point form equation of a line is

$$y - (-3) = 4(x - 2)$$

$\therefore \quad y + 3 = 4x - 8$

$\therefore \quad 4x - y - 8 - 3 = 0$

$\therefore \quad 4x - y - 11 = 0$

7) Find the equation of line passing through (4, 5) and perpendicular to the line $7x-5y-420=0$.

Solution : Given line is $7x - 5y - 420 = 0$

It's slope is $m_1 = \frac{-7}{-5} = \frac{7}{5}$

Let m_2 be the slope of required line, then

$$m_1 \cdot m_2 = -1 \quad \text{(condition of perpendicular line)}$$

$$\therefore \quad m_2 = \frac{-1}{\frac{7}{5}}$$

$$\therefore \quad m_2 = \frac{-5}{7}$$

Equation of required line having slope $\frac{-5}{7}$ and passing through point (4, 5) is given by

$$y - 5 = \frac{-5}{7}(x - 4)$$

$$7(y - 5) = -5(x - 4)$$

$$7y - 35 = -5x + 20$$

$$5x + 7y - 35 - 20 = 0$$

$\therefore \quad 5x + 7y - 55 = 0$

EXERCISE:

1) Show that the following pairs of lines are parallel.

1) $3x + y - 1 = 0,\ 21x + 7y - 2 = 0$
2) $2x + 3y + 7 = 0$ and $4x + 6y + 2 = 0$
3) $7x = 6 - 3y,\ 14x = 7 - 6y$

4) $5x + 4y - 4 = 0$, $15x + 12y + 10 = 0$
5) $x + y - 2 = 0$, $7x + 7y = 10$

2) Show that the following pairs of lines are perpendicular.

1) $5x + 6y - 1 = 0$, $6x - 5y + 3 = 0$
2) $2x + 3y = 5$, $2x - 3y = 6$
3) $2x - y + 1 = 0$, $x + 2y - 2 = 0$
4) $7x + y - 1 = 0$, $3x - 21y + 2 = 0$
5) $2x + 3y - 1 = 0$, $3x - 2y - 5 = 0$

3) Find the equation of the straight line

1) Passing through (2, 3) and parallel to the line $2x + 5y - 1 = 0$.
2) Parallel to $3x - 2y + 5 = 0$ and passing through the point (5, – 6).
3) Passing through (4, 5) and parallel to the line $2x - 3y - 5 = 0$.
4) Passing through (3, – 1) and parallel to the line $x + 2y - 4 = 0$.
5) Parallel to $x - 3y - 12 = 0$ and passing through (– 4, 2).

4) Find the equation of straight line

1) That passes through (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$.
2) Passing through (2, 0) and perpendicular to $x + y + 3 = 0$.
3) Through (4, – 5) and perpendicular to the line $3x + 4y + 5 = 0$.
4) Through (4, 5) and perpendicular to the line $3x + 5y - 20 = 0$.
5) Passing through (3, – 4) and perpendicular to $5x - 2y + 3 = 0$

5) Find the acute angle between the lines

1) $y = 5x + 6$ and $y = x$
2) $2y + x = 1$ and $x + 3y = 6$
3) $2x + 3y + 5 = 0$ and $x - 2y - 4 = 0$
4) $x + 3y + 5 = 0$ and $x - 2y - 4 = 0$
5) $x - 2y + 5 = 0$ and $7x + y - 10 = 0$
6) $2x + 3y = 13$ and $2x - 5y + 7 = 0$

6) If the slopes of two lines are $\frac{-5}{6}$ and $\frac{1}{11}$ then the measure of the angle between the lines.

7) Find the acute angle between the lines whose slopes are $\sqrt{3}$ and $\frac{1}{\sqrt{3}}$.

Perpendicular Distance between Point and Line:

If $P(x_1, y_1)$ is any point and $Ax + By + C = 0$ is a line, the perpendicular distance of a point P from the line is given by $\left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right|$.

Solved Examples

1) Find the length of the perpendicular on the line $3x + 4y - 5 = 0$ from the point (3, 4)

Solution : Given line is $3x + 4y - 5 = 0$

Here $A = 3$, $B = 4$, $C = -5$

Also, $P(x_1, y_1) = (3, 4)$

Length of perpendicular from $P(x_1, y_1)$ to the line $Ax + By + C = 0$ is given by

$$P = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right|$$

$$= \left|\frac{3(3) + 4\,(4) - 5}{\sqrt{(3)^2 + (4)^2}}\right|$$

$$= \left|\frac{9 + 16 - 5}{\sqrt{9 + 16}}\right| = \left|\frac{20}{\sqrt{25}}\right| = \left|\frac{20}{5}\right|$$

$\therefore$ $P = 4$ units.

2) Find the length of perpendicular from the point $(-3, -4)$ on the line $4\,(x + 2) = 3(y - 4)$.

Solution : Given line is $4(x + 2) = 3(y - 4)$

$$4x + 8 = 3y - 12$$

$$4x - 3y + 8 + 12 = 0$$

$$4x - 3y + 20 = 0$$

Here $A = 4, B = -3, C = 20$

Also, $P(x_1, y_1) = (-3, -4)$

Now length of perpendicular from $P(x_1, y_1)$ to the line $Ax + By + C = 0$ is given by

$$P = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right|$$

$$= \left|\frac{4(-3) + (-3)\,(-4) + 20}{\sqrt{(4)^2 + (-3)^2}}\right|$$

$$= \left|\frac{-12 + 12 + 20}{\sqrt{16 + 9}}\right| = \left|\frac{20}{\sqrt{25}}\right| = \left|\frac{20}{5}\right|$$

$\therefore$ $P = 4$ units.

Exercise:

Find the length of perpendicular from the given point on to the given line.

1) $(3, 4)$ on the line $3x + 4y = 7$.
2) $(3, -2)$ on the line $4x - 6y - 5 = 0$.
3) $(3, -2)$ on the line $6x - 4y - 5 = 0$.
4) $\left(0, \frac{5}{4}\right)$ on the line $6x + 8y - 45 = 0$.
5) $(1, -1)$ form $3x - 4y + 8 = 0$.
6) $(5, 4)$ on the line $2x + y + 6 = 0$.

Perpendicular Distance between Two Parallel Lines

The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by

$$d = \left|\frac{c_2 - c_1}{\sqrt{a^2 + b^2}}\right|$$

Solved Examples:

1) Find the distance between the parallel lines $3x - y + 7 = 0$ and $3x - y + 16 = 0$.

Solution : $3x - y + 7 = 0$, $3x - y + 16 = 0$

$$a = 3, \quad b = -1, \quad c_1 = 7, \; c_2 = 16$$

$\therefore$ Distance between two parallel lines is

$$d = \left|\frac{c_2 - c_1}{\sqrt{a^2 + b^2}}\right| = \left|\frac{16 - 7}{\sqrt{3^2 + (-1)^2}}\right|$$

$$= \left|\frac{9}{\sqrt{10}}\right|$$

$$d = \frac{9}{\sqrt{10}}$$

2) Find the distance between the lines, $3x + 4y + 5 = 0$ and $6x + 8y = 25$.

Solution : Given lines are

$$3x + 4y + 5 = 0$$
$$\therefore 2(3x + 4y + 5) = 2 \times 0$$
$$\therefore 6x + 8y + 10 = 0$$

$$6x + 8y = 25$$
$$\therefore 6x + 8y - 25 = 0$$

Here $a = 6$, $b = 8$, $c_1 = 10$, $c_2 = -25$

The distance between two parallel line is given by

$$\therefore \quad d = \left|\frac{c_2 - c_1}{\sqrt{a^2 + b^2}}\right|$$

$$= \left|\frac{-25 - 10}{\sqrt{(6)^2 + (8)^2}}\right| = \left|\frac{-35}{\sqrt{36 + 64}}\right| = \left|\frac{-35}{\sqrt{100}}\right|$$

$$\therefore \quad = \frac{35}{10}$$

$$d = \frac{7}{2} \text{ units}$$

Exercise:

1) Find the distance between parallel lines.
2) $5x - 12y + 1 = 0$ and $10x - 24y - 1 = 0$
3) $3x + 2y - 6 = 0$ and $3x + 2y - 12 = 0$.
4) $4x - 3y + 2 = 0$ and $4x + 3y - 9 = 0$.
5) $5x - 2\sqrt{6}\, y + 1 = 0$ and $5x - 2\sqrt{6}\, y - 10 = 0$
6) $5x - 12y + 1 = 0$ and $10x - 24y - 1 = 0$

Worksheet:

Unit 4
Mensuration

Course Outcome: Solve the problems based on measurement of regular closed figures and regular solids.

Unit outcome:

a) Calculate the area of given triangle and circle

b) Determine the area of the given square parallelogram, rhombus and trapezium.

c) compute surface area of given cuboids, sphere, cone and cylinder.

d) Determine volume of given cuboids sphere, cone and cylinder.

Introduction: Mensuration is the branch of mathematics which deals with the study of different geometrical shapes, their areas and Volumes. It is all about the process of measurement of 2 dimensional and 3 dimensional shapes.

Significance of Mensuration:

It is used to find area and volumes of various shapes

Area:

Area is the quantity that expresses the extent of two dimensional figure or shape, or plane lamina, in the plane.

Square meter (m^2), square centimeter (cm^2), square kilometer (km^2) are the units of area.

1) Area of any triangle

Let Δ denote the area of a triangle. Then area of a triangle is

i) $\Delta = \frac{1}{2}$ (Base) $\times$ (Height)

ii) $\Delta = \frac{1}{2}$ bc sinA etc, (i.e. Two sides and angles between them are given)

iii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Where a, b and c are sides of triangle and

$$s = \frac{a+b+c}{2} = \text{semi perimeter of triangle.}$$

2) Area of an equilateral triangle

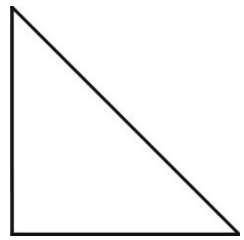
Let the sides of equilateral triangle be ‘a’ and for that triangle A = 60°.

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} \cdot a \cdot a \sin 60°$$

$$= \frac{1}{2} a^2 \frac{\sqrt{3}}{2}$$

$$\Delta = \frac{\sqrt{3}}{4} a^2$$

3) Area of right angle triangle



$$\Delta = \frac{1}{2}\ bc \cdot \sin A$$

$$= \frac{1}{2} bc \sin 90°$$

Where $A = 90°$

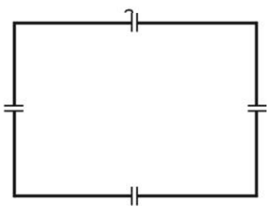
$$\boldsymbol{\Delta = \frac{1}{2} bc}$$

4) Area of a rectangle

Area = Δ = Length × Breadth

$$\boldsymbol{\Delta = a \times b}$$

5) Area of square

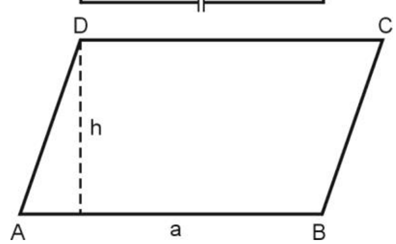


Here, Length = Breadth

$$\Delta = a \cdot b = a \cdot a = a^2$$

$$\boldsymbol{\Delta = a^2}$$

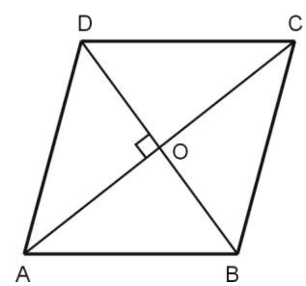
6) Area of parallelogram



$$\Delta = \text{Base} \times \text{Height}$$

$$\boldsymbol{\Delta} = a \cdot h$$

7) Area of a rhombus



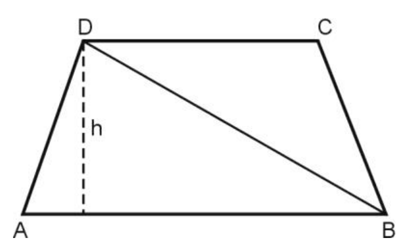
Rhombus has all sides equal and diagonals intersect at right angle.

$$\Delta = \frac{1}{2}\ AC \times BD$$

$$\Delta = \frac{1}{2}\ (\text{Product of diagonals})$$

8) Area of a trapezium / trapezoid

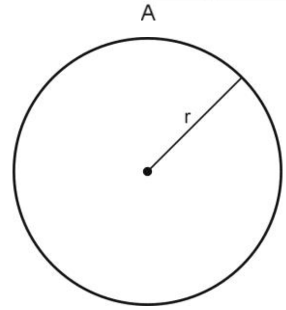
Trapezium is a quadrilateral in which two sides are parallel.



$$\text{Area} = \Delta = \frac{1}{2}\ [AB + CD] \cdot h$$

$$\Delta = \frac{1}{2}\ [\text{Sum of parallel sides}] \times \text{Height}$$

9) Area of a circle



The area of a circle having radius r is given by

$$\text{Area} = \pi r^2$$

Also, the circumference of a circle = $2\pi r$

OR πd (d-diameter)

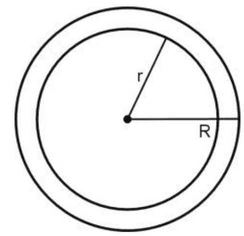
$$= \frac{1}{2}\ r^2\ [\theta - \sin\theta]$$

10)Area of annulas (Ring)

Let R, r be radius of outer and inner circles then area of annulas is equal to area of outer circle – area of inner circle.

i.e. $\pi R^2 - \pi r^2$

$= \pi(R^2 - r^2)$



Solved Problems:

1. Find the circumference and the area of the circle whose radius is 7.7 cm.

Solution : Hereradius= r = 7.7 cm

Area of the circle is given by

$$\therefore \quad A = \pi r^2 = \frac{22}{7}(7.7)^2$$

$$\mathbf{A = 186.34\ cm^2}$$

Circumference of the circle is

$$2\pi r = 2 \times \frac{22}{7} \times 7.7 = \mathbf{48.4\ cm}$$

2. What is the radius of a circle if its area is 120 cm^2 ? Also, find its circumference.

Solution:

Area=120 cm^2

Area of the circle is given by,

$$A = \pi r^2$$

$$120 = \frac{22}{7} r^2$$

$$\therefore \quad r^2 = \frac{7 \times 120}{22}$$

$$r^2 = 38.181$$

$$r = 6.18\ cm$$

Circumference of the circle is

$$2\pi r = 2 \times \frac{22}{7} \times 6.18 = 38.845\ cm$$

3. The area of rectangle with one side 8 cm is 172 cm^2. Find length of the other side.

Solution:

$$\text{Area} = l \times b$$

$$172 = 8 \times b$$

$$b = \frac{172}{8}$$

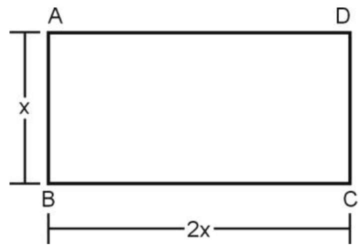
$$b = 21.5\ cm$$

Length of the other side is 21.5 cm

4. The length of one side of rectangle is twice the length of its adjacent side. If the perimeter of rectangle is 60 cms. Find the area of the rectangle.

Solution : Let, □ ABCD is a rectangle. In □ ABCD, $\overline{AB}$ and $\overline{BC}$ are adjacent sides.

Let, the length of $\overline{AB}$ = x cm



Then, the length of $\overline{BC}$ = 2x cm.

Also, the perimeter of rectangle is 60 cm.

$$\text{Perimeter} = \text{Sum of all sides}$$
$$60 = x + x + 2x + 2x$$
$$60 = 6x$$
$$\therefore \quad x = \frac{60}{6}$$
$$\therefore \quad \mathbf{x} = 10$$

Length of $\overline{AB}$ = x = 10 cm

Length of $\overline{BC}$ = 2x = 20 cm

Area of rectangle ABCD =Length × Breadth = 20 × 10 = 200 cm^2

The area of the given rectangle is 200 cm^2.

5. The area of a rectangular courtyard and is 3000 sq. m. Its sides are in the ratio 6:5. Find the perimeter of courtyard.

Solution

Area of rectangular courtyard. is = Length × Breadth

Given : $l : b = 6 : 5$

$$\text{i.e.} \quad \frac{l}{b} = \frac{6}{5}$$
$$\therefore \quad l = \frac{6}{5} b$$
$$\because \quad A = l \times b$$
$$3000 = \frac{6}{5} b \times b$$
$$\frac{15000}{6} = b^2$$
$$2500 = b^2$$
$$\therefore \quad b = 50$$
$$\therefore \quad l = \frac{6}{5} b = \frac{6}{5} \times 50$$
$$\therefore \quad l = 60$$

Perimeter of rectangular courtyard is = $2(l + b) = 2(60 + 50) = 220$ m.

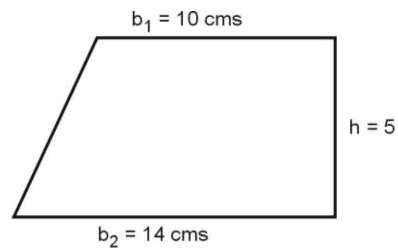
6. Find the area of a trapezoid with base of 10 cms and 14 cms and height of 5 cms.

Solution: The area of trapezoid is given by,

$$A = \frac{1}{2}(\text{Sum of Parallel side}) \times \text{Height}$$

$$= \frac{1}{2}(10 + 14) \times 5$$

$$\mathbf{A} = 60 \text{ cm}^2$$



7. Calculate the area of a rhombus whose diagonals are 30 and 16 cm.

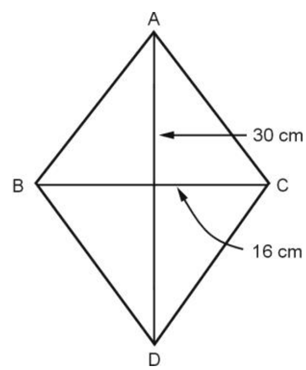
Solution : The area of rhombus is

$$A = \frac{1}{2}(AD \times BC)$$

$$= \frac{1}{2}(30 \times 16)$$

$$A = 240 \text{ cm}^2$$

$\therefore$ The area of □ ABCD is 240 cm^2.



Exercise:

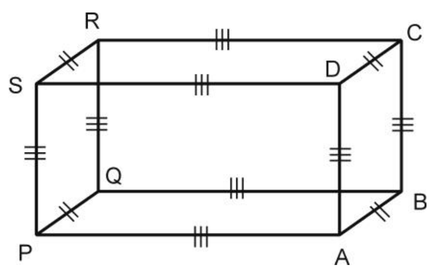
1. A circle has a diameter of 14 cm. Calculate its area.
2. In a right angle triangle, the length of one side is 4.5 cm and the length of the hypotenuse is 20.5 cm. Find the area of the right triangle.
3. A side of rhombus PQRS measures 8.5 units. If the length of the diagonal $\overline{QS}$ is 8 units. Find the area of a rhombus.
4. Find the area of a trapezium whose parallel sides are 10 cm and 8 cm, Where the perpendicular distance between the sides is 4 cm.
5. A wall is of the form of a trapezium with height 4 m and parallel sides being 3 m and 5 m. Find the cost of painting the wall if it has rate of painting as ` 25 per sq.m.
6. The area of a trapezoid is 24 sq.cm and the bases are 9 cm and 7 cm, Find the height.
7. Find area of rhombus whose diagonals are of length 10 cm and 8.2 cm.
8. Find the circumference of circle whose area is 38.5 cm^2
9. If diameter of a circle is 28 cm, find the area and circumference of the circle.
10. In a right triangle, the measure of one side is 24 cms and that of the hypotenuse is 25 cm. Find the area of the right triangle
11. In a rectangle, the length of one of its sides is half the length of its adjacent side. If the perimeter of rectangle of 43 units, find its area.

12. The two sides of a triangular park are 25 m and 60 m and angle between them is 30°. Find expenditure required to cut the grass if it costs ` 2/- per sq. m.
13. The area of the ring formed by two concentric circle is 346.5 cm^2. If the circumference of the inner circle is 88 cm. Then find out radius of the outer circle
14. Find the area of a farm in the form of a quadrilateral whose diagonal is 100 m offsets are 50 m and 30 m.

Volume and Surface Area

Cuboid

In everyday life, we come across so many objects like, a match box, a brick etc. Each of these objects is a cuboid in shape.



A body which has six faces, all rectangles, is called a cuboid or a rectangular solid.

1) Surface area:

Surface area of the cuboid = 2(l × b + b × h + h × l)

2) Volume of the cuboid

Volume of the cuboid = Length × Breadth × Height

= l × b × h

Solved Examples:

1) Find the surface area of a cuboid of dimensions 25 cm, 20 cm and 12 cm.

Solution :

Here *l* = 25 cm,b = 20 cm, h = 12 cm

Total surface area of cuboid (S_t)

= 2(lb + bh + hl)

= 2(25 × 20 + 20 × 12 + 12 × 25)

S_t = 2080 cm^2

Surface area of a cuboid is 2080 cm^2.

2) For a cuboid, the height, length and width are given by 13 cm, 35 cm and 22 cm respectively. Calculate its volume.

Solution : Here *l* = 35 cm, b = 22 cm, h = 13 cm

Volume of a cuboid (V) = l × b × h

= 35 × 22 × 13 = 10010 cm^3

Volume of a cuboid is 10010 cm^3.

3) Find the volume of the brick having size 30 cm by 25 cm by 10 cm.

Solution : Here $l = 30$ cm, b = 25 cm, h = 10 cm

Volume of a brick $= l \times b \times h$

$= 30 \times 25 \times 10 = 7500 \text{ cm}^3$

Volume of a brick is 7500 cm^3.

4) A match box measures 4 cm by 2.5 cm by 1.5 cm what will be the volume of packet containing 12 such match boxes ? How many such packets can be placed in a cardboard box whose size is 60 cm × 30 cm × 24 cm ?

Solution :

Volume of match box $= (4 \times 2.5 \times 1.5) = 15 \text{ cm}^3$

Volume of packet containing 12 match boxes

$= (12 \times 15) = 180 \text{ cm}^3$

Volume of cardboard box $= 60 \times 30 \times 24 = 43200 \text{ cm}^3$

$\therefore$ Number of packet that can be put in a cardboard $= \dfrac{43200}{180} = 240$

Exercise:

1) Find the surface area of cuboid whose sides are 3 cm by 6 cm by 10 cm.

2) The volume of a cuboid is 440 cm^3 and the area of its base is 88 cm^2. Find its height.

3) A box has length 36 cm, breadth 30 cm and height 40 cm. Find the cost of painting it from outside at the rate of 5 paise per cm^2.

4) The total surface area of a cuboidal cement concrete slab is 608 m^2. If the length of the slab is 30 m and height 10 cm, Find its breadth.

5) The outer dimensions of a closed wooden box are 10 cm by 8 cm by 7 cm. Thickness of the wood is 1 cm. Find the total cost of wood required to make box if 1 cm^3 of wood cost ` 2.00.

Cube

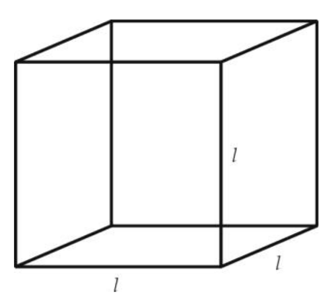
A cuboid whose length, breadth and height are equal to each other is called a cube.

Since, each face of a cube is a square : area of each face $(\text{side})^2 = l^2$

$\therefore$ Total surface area of the cube

$=$ Sum of the areas of its six faces

$= 6 \times (\text{side})^2 = 6 \times l^2$



Volume of the cube

$=$ Its length × breadth × height

$= l \times l \times l$

$= l^3$

1) Calculate the surface area of the cube having length of one side as 5.3 cm.

Solution : Given : $l = 5.3$ cm

To find surface area of the cube

Surface area of the cube $= 6l^2$

$= 6 \times (5.3)^2$

$= 168.54\ cm^2$

$\therefore$ Total surface area of the cube is 168.5 cm^2.

2) A cube having surface 96 cm^2. Find its edge length.

Solution : The surface area of cube is

$$6\,l^2 = 96$$

$$\therefore \quad l^2 = \frac{96}{6}$$

$$\therefore \quad l^2 = 16$$

$$\therefore \quad l^2 = 4\ cm.$$

So, the edge of length of cube is 4 cm.

3) The total surface area of a cube is 294 cm^2. Find the volume.

Solution : Here total area = $S_t = 294\ cm^2$

$$l = ?, \qquad V = ?$$

Total surface area of a cube $= 6\,l^2$

$$\therefore \quad 294 = 6\,l^2$$

$$l^2 = \frac{294}{6}$$

$$l^2 = 49$$

$$l = 7\ cm.$$

Volume of a cube $= l^3$

$= (7)^3 = 343\ cm^3$.

The volume of the cube is 343 cm^3.

4) The volume of a cube is 1,000 cm. Find its total surface area.

Solution : Let the Length of each edge of the cube be ' l ' cm. Then

Volume $= 1000\ cm^3$

$$l^3 = 1000$$

$$l = 10\ cm$$

$\therefore$ Surface area $= 6\,l^2$

$= 6 \times (10)^2 = 600\ cm^2$

5) Three cubes whose edges measure 3 cm, 4 cm, and 5 cm respectively to form a single cube. Find it edge. Also find the surface area of the new cube.

Solution : Let x cm be the edge of the new cube. Then volume of the new cube = Sum of the volume of three sides.

$$x^3 = 3^3 + 4^3 + 5^3 = 27 + 64 + 125$$
$$x^3 = 216$$
$$x = 6 \text{ cm}$$

$\therefore$ Edge of the new cube is 6 cm long.

Surface area of the new cube.

$$\therefore \quad 6x^2 = 6(6)^2 = 216 \text{ cm}^2$$

Exercise:

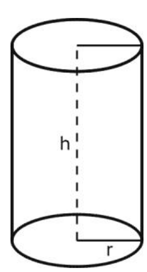
1) The side of a cube is 60 cm. Find the total surface area of the cube.
2) The diagonal of a cube is $\sqrt{12}$ cm. What is its edge ?
3) The perimeter of one face of a cube is 24 cm.
   Find i) The total area of the 6 faces.
   ii) The volume of the cube.
4) Two cubes, each with 12 cm edge, are joined end to end. Find the surface area of the resulting cuboid.
5) Find the volume of the cube. If the length of the one side is $\frac{3}{2}$ cm

Cylinder

1. Right Circular Cylinder

In our day to day life, we come across several solids like measuring jars, circular pillars, circular pipes, a garden roller, gas, cylinder etc. These solids have a curved (lateral) surface with congruent circular ends. Such solids are right circular cylinders.

A right circular cylinder has two plane ends. Each plane end is circular in shape, and the two plane ends are parallel; that is, they lie in parallel planes. Each of the plane end is called a base of the cylinder.

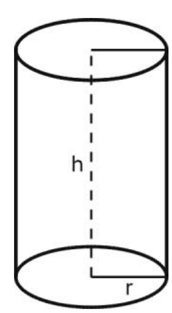


Volume of right circular cylinder

Here, the base is a circle of radius r and height is h

$\therefore$Volume = (Area of base) $\times$ Height

$$V = \pi r^2 h$$



Lateral surface

Lateral surface is a curved surface which is given by

$$s = 2\pi rh$$

and Total surface = $2\pi rh + 2\pi r^2 h$ (base area)

Solved Examples:

1) Find the volume of a cylinder having radius 7 cm and height 12 cm.

Solution : Given Radius $= r = 7$ cm

and Height $= h = 12$ cm

To find Volume = ?

$$\text{Volume of cylinder} = \pi r^2 h$$

$$= \frac{22}{7} \times (7)^2 \times 12 = 22 \times 7 \times 12$$

$$\text{Volume} = 1848 \text{ cm}^3$$

$\therefore$ The volume of the cylinder is 1848 cm^3

2) The area of the base of a right circular cylinder is 154 cm^2 and its height is 15 cm. Find the volume of the cylinder.

Solution : We have

$$\text{Volume} = (\text{Area of base}) \times \text{Height}$$

$$= 154 \times 15 = 2310 \text{ cm}^3$$

$\therefore$ Volume of the cylinder = 2310 cm^3

3) The circumference of the base of a cylinder is 132 cm and its height 25 cm. Find the volume of the cylinder.

Solution : Let r cm be the radius of the cylinder.

Then Circumference = 132 cm

$$2\pi r = 132$$

$$2 \times \frac{22}{7} \times r = 132$$

$$r = \frac{132 \times 7}{2 \times 22} = 21 \text{ cm}$$

We have Height $= h = 25$ cm

$$\text{Volume of the cylinder} = \pi r^2 h = \frac{22}{7} \times (21)^2 \times 25$$

$$\text{Volume of the cylinder} = 34650 \text{ cm}^3$$

4) Find the capacity of a cylindrical water tank whose radius is 2.1 m and height is 5 m.

Solution :

Here the radius of a cylindrical water tank is 2.1 m and height is 5 m.

$$\text{Volume of tank} = \pi r^2 h$$

$$= 3.14 \times (2.1)^2 \times 5 = 69.3 \text{ cm}^3$$

$\therefore$ Volume of tank is 69.3 cm^3

We know that $1 \text{ m}^3 = 1000$ litre

$\therefore$ $69.3 \text{ m}^3 = 69.3 \times 1000 = 69300$ litre

$\therefore$ The capacity of a cylindrical water tank is 69300 litre.

5) Find the curved surface area and total surface area of a right circular cylinder whose height is 15 cm and the radius of the base is

7 cm. $\left(\text{Take } \pi = \frac{22}{7}\right)$

Solution : Here r =Radius = 7 cm

h = Height = 15 cm

Curved surface area of the cylinder $= 2\pi rh$

$$= 2 \times \frac{22}{7} \times 7 \times 15$$

$$= 660 \text{ cm}^2$$

Total surface area of the cylinder $= 2\pi r(h + r)$

$$= 2 \times \frac{22}{7} \times 7 \times (15 \times 7) \text{ cm}^2$$

$$= 968 \text{ cm}^2$$

6) Find the height of a cylinder whose radius is 7 cm and the total surface area is 968 cm^2.

Solution : Let Height of the cylinder be h cm.

Given r = 7 cm and

Total surface area = 968 cm^2

$$2\pi r (h + r) = 968$$

$$2 \times \frac{22}{7} \times 7 \times (h + 7) = 968$$

$$\therefore \quad 44(h + 7) = 968$$

$$h + 7 = \frac{968}{44}$$

$$h + 7 = 22$$

$$h = 15$$

Hence, the height of the cylinder is 15 cm.

EXERCISE:

1) A cylinder has radius 6.7 cm and height 9 cm, Find the volume of the cylinder.
2) The volume of a cylinder is 448π cm^3 and height 7 cm. Find its lateral surface area and total surface area.
3) Find the volume of a right circular cylinder which has a height of 21 cm and the base radius 5 cm. Find also the curved surface area of the cylinder.
4) The whole surface of a right circular cylinder is 7 sq.ft., 37 sq.in and the diameter of the base is half the height. Find the height.
5) The volume of a right cylinder is 1100 cu. cms and the radius of its base 5 cms. Find the area of the curved surface.

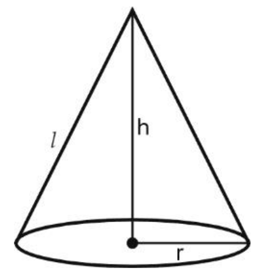
Cone : A cone is a solid generated by a line, one end of which is fixed and the other end describes a closed curve in a plane. The fixed point is called the vertex or apex.

If the base is a circle, the cone is called circular cone. The line joining the vertex of the cone with the centre of the circle (base) is called "axis" of the cone. If the axis of the cone is perpendicular to the base, then, the cone is said to be "right circular cone".

Volume of a right circular cone

$$V = \frac{1}{3}(\text{area of base}) \times \text{height}$$

$$V = \frac{1}{3}(\pi r^2) \cdot h$$



Curved surface

$$= \frac{1}{2}(\text{perimeter of the base}) \times \text{slant height}$$

$$= \frac{1}{2}(2\pi r) \cdot l = \pi r l$$

Where Slant height (l) = $\sqrt{h^2 + r^2}$

and r is the radius of circular base.

Total surface area

$$= \pi r l + \pi r^2$$

$$= \pi r(r + l)$$

Solved Examples:

1) Find the volume of a cone having radius 10 cm and height 20 cm.

Solution : The volume of a cone = $\frac{1}{3}\pi r^2 h$

Where r = radius = 10 cm

h = Height = 20 cm

$$\text{Volume} = \frac{1}{3} \cdot \frac{22}{7} \cdot (10)^2 \cdot 20$$

$$\text{Volume} = 2095.23 \text{ cm}^3$$

2) The height and slant height of a cone are 12 cm and 20 cm respectively. Find its volume. (Take $\pi = 3.14$)

Solution : We have a cone having height 12 cm and slant height 20 cm.

First we have find the radius of a cone.

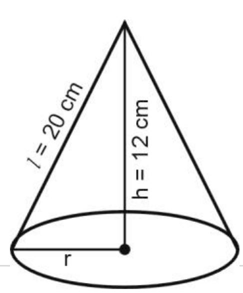
By Pythagoras theorem

$$r^2 + h^2 = l^2$$

$$r^2 = l^2 - h^2$$

$$= (20)^2 - (12)^2$$

$$r^2 = 256$$



$\therefore \quad r = 16$ cm.

The radius of the cone is 16 cm.

Now volume of a cone

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times 3.14 \times (16)^2 \times 12 = 3215.36 \text{ cm}^2$$

$\therefore$ Volume of a cone is 3215.36 cm^2.

3) A cone has a circular base of radius 10 cm and slant height of 30 cm. Calculate the surface area.

Solution :

$$\text{Radius of cone} = 10 \text{ cm} = r$$
$$\text{Slant height} = 30 \text{ cm} = l$$
$$\text{Total surface area} = \pi r (r + l)$$
$$= \frac{22}{7} \times 10(10 + 30)$$
$$= 1257.14 \text{ sq.cm}$$

4) A right pyramid of height 12 cms stands on square base whose side is 10 cms. Find 1) Slant height 2) Slant surface 3) Volume of the pyramid.

Solution :

1) Slant height $= \sqrt{(12)^2 + (5)^2} = \sqrt{169} = 13$

$\therefore$ Slant height = 13 cms.

2) Slant surface $= \frac{1}{2}$ (Perimeter of base) $\times$ Slant height

$$= \frac{1}{2}(10 + 10 + 10 + 10) \times (13)$$
$$= 260 \text{ sq.cm.}$$

3) Volume $= \frac{1}{3}$ (Area of base) $\times$ Height

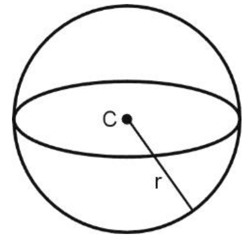
$$= \frac{1}{3}(100) \times 12 = 400 \text{ cm}^3$$

EXERCISE:

1) The radius of the base of a right circular cone is 6 cms and the slant height is 6.5 cms. Find the volume.
2) The curved surface area of a cone is 4070 cm^2 and its diameter is 70 cm. What is its slant height ? $\left(\text{Use } \pi = \frac{22}{7}\right)$
3) Find the volume of a right circular cone whose diameter is 6 cm and slant height 5 cm.
4) Find the curved surface area of a cone having diameter 2 m and height $4\sqrt{3}$ m.
5) The height of a cone is 2.5 cm and the radius of its base 6 cm. Find its area.
6) If the circumference is 14 cm and height is 11 cm. Find the volume of cone.

Sphere

A sphere is a solid bounded by a curved surface every point of which is equidistance from a point within it (called as centre). The distance of any point of the surface from the centre is called 'radius of the sphere'.



The volume of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^2$$

And Surface area = $S = 4\pi r^2$

Solved Examples:

1) Find the volume and surface area of a sphere of radius 4.2 cm. $\left(\text{Take } \pi = \frac{22}{7}\right)$.

Solution : We have

$$r = \text{Radius of the sphere} = 4.2 \text{ cm}$$

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \frac{22}{7} \times (4.2)^3$$

$$= 310.464 \text{ cm}^3$$

$$\text{Surface area of the sphere} = \frac{4}{3}\pi r^2$$

$$= 4 \times \frac{22}{7} \times (4.2)^2$$

$$= 221.76 \text{ cm}^2$$

2) Total volume of 21 steel balls in a bearing is 88 cubic cm. Find the diameter of each ball.

Solution :

$$\therefore \quad \text{Volume of steel ball} = \frac{4}{3}\pi r^3$$

$$\frac{88}{21} = \frac{4}{3}\pi r^3$$

$$\frac{7}{22} \times \frac{88}{21} \times \frac{3}{4} = r^3$$

$$\therefore \quad r^3 = 1$$

$$\therefore \quad r = 1 \text{ cm}$$

$$\therefore \text{Diameter of steel ball} = 2r$$

$$= 2(1) = 2 \text{ cm.}$$

3) The surface area of the sphere is 616 sq.cm. Find the diameter of the sphere.

Solution : We know that the surface area of a sphere. = $4\pi r^2$

$$\therefore \quad 616 = 4\pi r^2$$

$$\frac{616 \times 7}{4 \times 22} = r^2$$

$$\therefore \quad r^2 = 49$$

$$\therefore \quad r = 7 \text{ cm}$$

$\therefore$ The radius of the sphere = 7 cm

Diameter (d) = 2r = 14 cm.

4) The volumes of two spheres are in the ratio 64 : 27. Find their radii if the sum of their radii is 21 cm.

Solution : Let the radii of two spheres be r_1 and r_2 respectively. Let the volume of two spheres be V_1 and V_2 respectively.

$$\therefore \quad \frac{V_1}{V_2} = \frac{64}{27}$$

$$\therefore \quad \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{64}{27}$$

$$\therefore \quad \frac{r_1^3}{r_2^3} = \left(\frac{4}{3}\right)^3$$

$$\therefore \quad \frac{r_1}{r_2} = \frac{4}{3}$$

$$\therefore \quad r_1 = \frac{4r_2}{3}$$

Given $\quad r_1 + r_2 = 21$

$$\frac{4\,r_2}{3} + r_2 = 21$$

$$\frac{7\,r_2}{3} = 21$$

$$7\,r_2 = 63$$

$$\boxed{r_2 = 9 \text{ cm}}$$

$$r_1 = \frac{4\,r_2}{3} = \frac{4(9)}{3}$$

$$= \frac{36}{3} = 12 \text{ cm.}$$

Hence, the radii of two spheres are 12 cm and 9 cm.

5) A lead bar 10 × 5 × 4 cms is melted and 5 equal spherical bullets are made. Find the diameter of the bullet.

Solution : Volume of the bar = 10 × 5 × 4 = 200 cu.cms

Let 'r' be the radius of each bullet.

Then $\quad 5 \times \frac{4}{3}\pi r^3 = 200$

$$r^3 = \frac{200 \times 3}{5 \times 4 \times \pi}$$

$$r^3 = \frac{30}{\pi} \qquad \therefore \quad r^3 = \frac{30 \times 7}{22} = \frac{105}{11} = 9.55$$

$$r = \sqrt[3]{9.55}$$

So that the diameter of each bullet is $2\sqrt[3]{9.55}$ cm.

EXERCISE:

1) If the volume of a sphere is $\frac{4\pi}{3}$ cc. Find its surface area.
2) The volume of a sphere is $\frac{88}{21}$ cubic metres. Find its surface area.
3) Find the volume and total surface area of a hemisphere of radius 3.5 cm $\left(\text{Use } \pi = \frac{22}{7}\right)$.
4) The surface area of the sphere is 616 cm^2. Find the diameter of the sphere.
5) A solid metallic cylinder of radius 18 cms and height $\frac{16}{3}$ cms. is melted and spherical balls of diameter 12 cms are made out of it. Find the number of balls made.
6) The volume of two spheres are in the ratio 27 : 64. Find their radii if the sum of their radii is 28 cm.
7) A solid right circular cone of radius 2 m and height 27 m is melted and recasted into a sphere. Find the volume and surface area of the sphere.

Worksheet:

Unit 5
Statistics

Course Outcome: Use basic concepts of statistics to solve engineering related problems

Unit outcome:

a) Obtain range and co-efficient of range of the given grouped and ungrouped data
b) Calculate mean and standard deviation of discrete and grouped data related to given simple engineering problems
c) Determine variance and co-efficient of variance of given grouped and ungrouped data

Introduction: Statistics is a science of collection, presentation , analysis and interpretation of the numerical data which is very essential for development of modern statistical techniques used in engineering fields.

Significance :

Comparison of statistical data leads us to make better decision.

Range:

Range for raw data:

Raw data: Range for raw data is defined as the difference between the smallest value and largest value in the given data .

L = Largest value, S = Smallest value

Range = L – S

Co-efficient of range : = $\frac{L-S}{L+S}$

Solved Example:

1) Find the range and coefficient of range for the following data.

200, 210 208 , 160 , 250 , 290

Solution: 200, 210, 208, 160, 250, 290

Smallest value = S = 160

Largest value = L = 290

Range = largest value – Smallest value

= L – S

= 290 – 160

Range = 130

Co-efficient of range $= \frac{L-S}{L+S}$

$= \frac{290-160}{290+160}$

$= \frac{130}{450} = 0.28889$

Exercise:

Find the range and coefficient of range of the following distribution

1) 15, 25, 35, 45, 55, 65
2) 45, 42, 39, 40, 48, 41, 45, 44.
3) 10, 5, 12, 2, 15, 20, 8, 10
4) 50, 90, 120, 9, 13, 11, 5

Range for ungrouped data:- Range for ungrouped data is defined as the difference between the smallest value of x_i and largest value of x_i in the given data.

L = Largest value of x_i, S = Smallest value of x_i

Range = L – S

Co-efficient of range: = $\frac{L-S}{L+S}$

Solved Example:

1) Find the range and coefficient of range for the following data .

x_i	5	10	15	20	25	30	35	40
f_i	2	3	7	5	7	8	8	10

Solution

x_i	**5**	10	15	20	25	30	35	40
f_i	2	3	7	5	7	8	8	10

Smallest value of x_i = S = 5

Largest value of x_i = L = 40

Range = Smallest value of x_i - Largest value of x_i

= L - S

= 40 -5

Range = 35

Co-efficient of range = $\frac{L-S}{L+S}$

$= \frac{40-5}{40+5}$

$= \frac{35}{45} = 0.77778$

Exercise:

1) Find range and coefficient of range of the following distribution

$\mathbf{x_i}$	10	20	30	40	50
f_i	7	5	3	2	1

2) Find range and coefficient of range of the following distribution

x_i	3	8	13	18	23	28	33
f_i	1	4	5	7	2	3	10

3) From the following data, calculate (i) Range, (ii) Coefficient of range

Mark	5	15	25	35	45	55
No. of students	10	20	30	40	50	60

Range for grouped data: - Range for grouped data is defined as the difference between upper boundary of highest class and lower boundary of lowest class in the given grouped data .

U = upper boundary of highest class,

L = lower boundary of lowest first class

Range = U – L

co-efficient of range = $\frac{U-L}{U+L}$

Solved Examples:

1) Find the range and coefficient of range for the following data

Marks	10 - 20	20-30	30-40	40-50	50 - 60
No. of students	6	19	34	10	18

Solution:

x_i	10-20	20-30	30-40	40-50	50-60
f_i	6	19	34	10	18

U = upper boundary of last class ,=60

L = lower boundary of first class =10

Range = U – L

= 60-10

= 50

co-efficient of range = $\frac{U-L}{U+L}$

$= \frac{60-10}{60+10}$

$= \frac{50}{70}$

$= 0.71429$

2) Find the range and coefficient of range for the following data

Marks	10-19	20-29	30-39	40-49	50-59	60-69
No.of students	6	10	16	14	8	4

Solution : Given data is discontinuous data , so we first find the class boundaries

Xi	10-19	20-29	30-39	40-49	50-59	60-69

C.B	9.5 -19.5	19.5-29.5	29.5-39.5	39.5-49.5	49.5-59.5	59.5- 69.5
Fi	6	10	16	14	8	4

U = upper boundary of highest class ,=69.5

L = lower boundary of lowest class =9.5

Range = U – L

= 69.5 – 9.5

= 60

$$\text{co-efficient of range} = \frac{U-L}{U+L}$$

$$= \frac{69.5-9.5}{69.5+9.5}$$

$$= \frac{60}{79}$$

$$= 0.76$$

Exercise:

1) From the following data, calculate (i) Range and (ii) Coefficient of range

Mark	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	10	20	15	25	30	25

2) Find the range and coefficient of range of the distribution

Class interval	11-20	21-30	31-40	41-50	51-60
Frequency	09	12	16	14	10

3) Calculate the range and coefficient of range for the following data :

Class :	21 - 25	26 - 30	31 - 35	36 - 40	41 - 45
Frequency :	4	16	38	12	10

4) Find range and coefficient of range for the following

Mark	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
No.of students	10	15	16	20	21	22	09	08

5) The weight of the students is given below. Calculate the range and coefficient of range for the same.

Weight (kg)	60 - 62	63 - 65	66 - 68	69 - 71	72 - 74
No. of students	5	18	42	27	8

Mean Deviation

Mean Deviation for raw data

i) Mean deviation about mean = $\frac{\Sigma|x_i - \overline{x}|}{N} = \frac{\Sigma\,|d_i|}{N}$

where $\overline{x}$ = mean of N observations.

Solved Example:

1) Calculate the mean deviation about the mean of the following data.

   3, 6, 5, 7, 10, 12, 15, 18

Solution : Given data is raw data

$$\text{Mean} = \bar{x} = \frac{\Sigma x_i}{N} = \frac{3+6+5+7+10+12+15+18}{8}$$

$$= \frac{76}{8}$$

$$= 9.5$$

x_i	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$
3	6.5
6	3.5
5	4.5
7	2.5
10	0.5
12	2.5
15	5.5
18	8.5
	$\Sigma \lvert d_i \rvert = 34$

$$\text{M.D.} = \frac{\Sigma \lvert d_i \rvert}{N} = \frac{34}{8}$$

$$\text{M.D.} = 4.25$$

Exercise:

Determine the mean deviation from mean of the following data

1) 12, 6, 7, 3, 15, 10, 18, 5
2) 34, 32, 18, 20, 28, 15, 17, 22, 25, 29
3) 30, 35, 40, 42, 38, 27, 31, 36, 40, 41
4) 5, 9, 1, 7, 3, 8, 6, 2, 4

Mean Deviation for Discrete (ungrouped) Frequency Distribution

$$\text{M.D. about mean} = \frac{\sum f_i \lvert x_i - \bar{x} \rvert}{\sum f_i} = \frac{\sum f_i \lvert d_i \rvert}{N}$$

where $N = \sum f_i$ and $\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$

Solved Example:

1) Calculate the Mean deviation from Mean of the following data.

x_i	10	11	12	13	14
f_i	3	12	18	12	3

Solution: Mean deviation about mean :

x_i	f_i	$f_i\,x_i$	$\|d_i\| = \|x_i - \bar{x}\|$	$f_i\,\|d_i\|$
10	3	30	2	6
11	12	132	1	12
12	18	216	0	0
13	12	156	1	12
14	03	42	2	06
	$N = \sum f_i = 48$	$\sum f_i x_i = 576$		$\sum f_i\,\|d_i\| = 36$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{N} = \frac{576}{48} = 12$$

$$\text{M.D.} = \frac{\sum f_i |d_i|}{N} = \frac{36}{48} = 0.75$$

Exercise:

1) Calculate the mean deviation about the mean of the following distribution.

x_i	3	4	5	6	7	8
f_i	4	9	10	8	6	3

2) Calculate the mean deviation about mean from the data.

x	3	9	17	23	27
v	8	10	12	9	5

4) Find the mean deviation from the mean for the following frequency distribution.

x_i	8	16	24	32	40	48	56	64
f_i	5	13	12	8	6	10	9	3

Mean deviation for grouped data :

$$\text{M.D.} = \frac{\sum f_i |x_i - \bar{x}|}{\sum f_i} = \frac{\sum f_i |d_i|}{N}$$

where x_i = Mid – Value or centre value

$\bar{x}$ = Mean

$N = \sum f_i$

Solved Examples:

1) Find the mean deviation from mean of the following distribution.

Marks	0-10	10-20	20-30	30-40	40-50
No. of students	5	8	15	16	6

Solution

C.I.	f_i	x_i	$f_i\,x_i$	$\lvert d_i\rvert = \lvert x_i - \bar{x}\rvert$	$f_i\,\lvert d_i\rvert$
0-10	5	5	25	22	110
10-20	8	15	120	12	96
20-30	15	25	375	2	30
30-40	16	35	560	8	128
40-50	6	45	270	18	108
	$N = \sum f_i = 50$		$\sum f_i x_i = 1350$		$\sum f_i \lvert d_i\rvert = 472$

$$\text{Mean} = \bar{x} = \frac{\sum f_i\, x_i}{\sum f_i}$$

$$= \frac{1350}{50} = 27$$

$$\text{Mean Deviation about mean} = \frac{\sum f_i \lvert d_i \rvert}{N}$$

$$= \frac{472}{50}$$

$$= 9.44$$

2) Calculate the mean deviation for the following data :

Class intervals	40-59	60-79	80-99	100-119	120-139
No. of families	50	300	500	200	60

Solution:

Class	Continue class	x_i	f_i	$f_i\,x_i$	$\lvert d_i\rvert = \lvert x_i - \bar{x}\rvert$	$f_i\,\lvert d_i\rvert$
40-59	39.5-59.5	49.5	50	2475	38.559	1927.95
60-79	59.5-79.5	69.5	300	20850	18.559	5567.7
80-99	79.5-99.5	89.5	500	44750	1.441	720.5
100-119	99.5-119.5	109.5	200	21900	21.441	4288.2
120-139	119.5-139.5	129.5	60	7770	41.441	2486.86
			$N = \sum f_i = 1110$	$\sum f_i x_i = 97745$		$\sum f_i \lvert d_i\rvert = 14990.81$

$$\text{Mean} = \bar{x} = \frac{\sum f_i\, x_i}{N} = \frac{97745}{1110} = 88.059$$

$$\text{M.D.} = \frac{\sum f_i \lvert d_i \rvert}{N} = \frac{14990.81}{1110} = 13.505$$

Exercise:

1) Find the mean deviation from the mean.

Class interval	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	4	6	10	18	9	3

2) Compute the mean deviation from mean for the following frequency distribution.

Production of Chikoos (in quintals)	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36	36-40
Number of Chikoo trees	5	8	18	25	15	12	10	5	2

3) Calculate mean deviation from the mean

Age in year	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No. of person	4	6	10	20	10	6	4

4) Find the mean deviation from mean for the following frequency distribution

Class	0-10	10-20	20-30	30-40	40-50
Frequency	1	2	4	2	1

Standard deviation (S.D.)

Standard deviation for raw data :

$$\text{S.D.} = \sigma = \sqrt{\frac{\sum\left(x_i - \bar{x}\right)^2}{N}} = \sqrt{\frac{\sum d^2_i}{N}}$$

$\bar{x}$ = Mean

$d_i = |x_i - \bar{x}|$

N = Total number of observation

1) Calculate S.D. of the following data :

25, 50, 30, 70, 42, 36, 48, 34, 60

Solution : Given data is raw data

$$\bar{x} = \frac{\sum x_i}{N} = \frac{25+50+30+70+42+36+48+34+60}{9} = \frac{395}{9} = 43.888$$

x_i	$d_i = x_i - \bar{x}$	d_i^2
25	–18.88	356.45
50	6.12	37.45
30	–13.88	192.65
70	26.12	682.25
42	– 1.88	3.53
36	– 7.88	62.09
48	4.12	16.97
34	– 9.88	97.614
60	16.12	259.85
		$\Sigma d_i^2 = 1708.85$

Standard Deviation $= \sqrt{\frac{\Sigma d^2_i}{N}}$

$= \sqrt{\frac{1708.85}{9}}$

$= \sqrt{189.872}$

$= 13.779$

Exercise:

Find standard deviation of the following

1) 1, 2, 3, 4, 5, 6, 7, 8, 9.
2) 6, 7, 10, 12, 13, 4, 8, 12
3) 69, 67, 68, 66, 69, 64, 63, 65, 72
4) 60, 45, 48, 52, 65, 50, 61, 59, 51, 49

Standard deviation for ungrouped data :

$$S.D. = \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}}$$

where $N = \Sigma f_i$

$\bar{x} = \text{Mean} = \frac{\Sigma f_i x_i}{N}$

$d_i = (x_i - \bar{x})$

Solved Example:

1) Find the standard deviation of the following frequency distribution.

x_i	6	7	8	9	10	11	12
f_i	3	6	9	13	8	5	4

Solution :

x_i	f_i	$f_i\,x_i$	$d_i = x_i - \bar{x}$	d_i^2	$f_i\,d_1^2$
6	3	18	–3	9	27
7	6	42	–2	4	24
8	9	72	–1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	$N = \sum f_i = 48$	$\sum f_i x_i = 432$			$\sum f_i d_i^2 = 124$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{N}$$

$$= \frac{432}{48}$$

$$\bar{x} = 9$$

$$\text{S.D.} = \sqrt{\frac{\sum f_i d_i^2}{N}}$$

$$= \sqrt{\frac{124}{48}}$$

$$= \sqrt{2.583}$$

$$\text{S. D.} = 1.607$$

Exercise:

1. Find arithmetic mean and S.D. of the following distribution

Age in year	25	35	45	55	65	75	85
No. of workers	3	61	132	153	140	51	2

2. Find the standard deviation of the following frequency distribution.

x_i	2	4	6	8	10	12	14	16
f_i	4	4	5	15	8	5	4	5

3. Find the standard deviation of the following frequency distribution.

$\mathbf{x_i}$	60	61	62	63	64	65	66	67	68
$\mathbf{f_i}$	2	1	2	29	25	12	10	4	5

Standard deviation for grouped data

$$\text{S.D.} = \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}}$$

where, $N = \Sigma f_i$

$d_i = | x_i - \bar{x} |$

$$\bar{x} = \text{Mean} = \frac{\Sigma f_i x_i}{N}$$

x_i = Mid - Value

Solved Examples:

1) Find Standard deviation of the following data

C.I.	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Frequency	3	5	8	3	1

Solution :

C.I.	f_i	x_i	$f_i x_i$	$d_i = \| x_i - \bar{x} \|$	d_i^2	$f_i d_i^2$
0 - 10	3	5	15	17	289	867
10 - 20	5	15	75	7	49	245
20 - 30	8	25	200	3	9	72
30 - 40	3	35	105	13	169	507
40 - 50	1	45	45	23	529	529
	$N = \Sigma f_i = 20$		$\Sigma f_i x_i = 440$			$\Sigma f_i d_i^2 = 2220$

$$\text{Mean} = \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{440}{20} = 22$$

$$\text{Standard Deviation .} = \sqrt{\frac{\Sigma f_i d_i^2}{N}}$$

$$= \sqrt{\frac{2220}{20}} = \sqrt{111}$$

Standard Deviation. = 10.53

2) Calculate Mean and Standard deviation for the following data.

Class	20-29	30-39	40-49	50-59	60-69	70-79
Frequency	10	15	30	20	15	10

Class	**Cont. Class**	f_i	x_i	$f_i x_i$	$d_i = \| x_i - \bar{x} \|$	d_i^2	$f_i d_i^2$
20-29	19.5-29.5	10	24.5	245	24.5	600.25	6002.5
30-39	29.5-39.5	15	34.5	517.5	14.5	210.25	3153.75
40-49	39.5-49.5	30	44.5	1335	4.5	20.25	607.5
50-59	49.5-59.5	20	54.5	1090	5.5	30.25	605
60-69	59.5-69.5	15	64.5	967.5	15.5	240.25	3603.75

70-79	69.5-79.5	10	74.5	745	25.5	650.25	6502.5
		$N = \sum f_i = 100$		$\sum f_i x_i = 4900$			$\sum f_i d_i^2 = 20475$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{N}$$

$$= \frac{4900}{100}$$

$$= 49$$

Standard Deviation $= \sigma$

$$= \sqrt{\frac{\sum f_i d_1^2}{N}}$$

$$= \sqrt{\frac{20475}{100}}$$

$$= 14.309$$

Exercise:

1. Calculate the mean and standard deviation for the following data

Expenditure in `	0-10	10-20	20-30	30-40	40-50
Frequency	14	23	27	21	15

2. Calculate the mean and standard deviation for the following frequency distribution.

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

3. The crushing strength of 45 cement blocks is recorded as

Crushing strength in kg/cm^2	145-155	155-165	165-175	175-185	185-195	195-205
No. of Blocks	6	7	9	14	4	5

find standard deviation

4. The following table shows the chest measurement of 100 students. Calculate the mean and standard deviation.

Chest in cm	67-74	75-81	82-88	89-95	96-102	103-109
Number of students	5	31	40	20	3	1

Variance-

The square of standard deviation is called the variance

Raw data

$$\text{Variance} = (\text{S.D.})^2$$

$$\bar{x} = \text{Mean} = \frac{\Sigma x_i}{N}$$

N = Total number of observations

$$\text{Coefficient of variance} = \frac{\text{S.D.}}{\text{Mean}} \times 100$$

$$= \frac{\sigma}{\bar{x}} \times 100$$

Solved Example:

1) Find variance and coefficient of variance of the following data

1) 49, 63, 46, 59, 65, 52, 60, 54

Solution

$$\bar{x} = \frac{\Sigma x_i}{N}$$

$$= \frac{49 + 63 + 46 + 59 + 65 + 52 + 60 + 54}{8}$$

$$\bar{x} = \frac{448}{8} \qquad \therefore \bar{x} = 56$$

x_i	$d_i = x_i - \bar{x}$	d_i^2
49	– 7	49
63	7	49
46	– 10	100
59	3	9
65	9	81
52	– 4	16
60	4	16
54	– 2	4
		$\Sigma d_i^2 = 324$

$$\text{S.D.} = \sigma = \sqrt{\frac{\Sigma d_i^2}{N}} = \sqrt{\frac{324}{8}}$$

S.D. = σ = 6.363

$$\text{Variance} = (\text{S.D.})^2 = (6.363)^2$$

= 40.487

$$\text{Coefficient of variance} = \frac{\text{S.D.}}{\text{Mean}} \times 100$$

$$= \frac{\sigma}{\bar{x}} \times 100 = \frac{6.363}{56} \times 100$$

= 11.362 %

Exercise:

Find variance and coefficient of variance of following data

1) 1, 2, 3, 4, 5, 6, 7
2) 25, 50, 30, 70, 42, 36, 48, 34, 60

Variance for ungroup data

Solved Examples:

1) Calculate variance for the data

x	10	20	30	40	50
f	12	15	17	11	9

Solution :

x_i	f_i	$f_i x_i$	$d_i = (x_i - \bar{x})$	d_i^2	$f_i d_i^2$
10	12	120	– 18.437	339.92	4079.04
20	15	300	– 8.437	71.182	1067.73
30	17	510	1.563	2.442	41.541
40	11	440	11.563	133.70	1470.7
50	9	450	21.563	464.96	4184.64
	$N = \Sigma f_i = 64$	$\Sigma f_i x_i = 1820$			$\Sigma f_i d_i^2 = 10843.65$

$$\text{Mean} = \bar{x} = \frac{1820}{64}$$

$$\mathbf{\bar{x} = 28.437}$$

$$\text{S.D.} = \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}}$$

$$= \sqrt{\frac{10843.65}{64}}$$

$$\textbf{S.D.} = \sigma = \mathbf{13.016}$$

$$\text{Variance} = (\text{S.D.})^2$$

$$= (13.016)^2$$

$$\textbf{Variance} = \mathbf{169.416}$$

Exercise:

1. Find variance and co-efficient of variance of following data

x	10	15	20	25
f	17	22	19	16

Variance for grouped data :

Solved Examples:

1) Find variance and coefficient of variance of the following data.

C.I.	0-10	10-20	20-30	30-40	40-50
Frequency	14	23	27	21	15

Solution:

Class	f_i	x_i	$f_i x_i$	$d_i = \lvert x_i - \bar{x} \rvert$	d_i^2	$f_i d_i^2$
0-10	14	5	70	20	400	5600
10-20	23	15	345	10	100	2300
20-30	27	25	675	0	0	0
30-40	21	35	735	10	100	2100
40-50	15	45	675	20	400	6000
	$N = \Sigma f_i = 100$		$\Sigma f_i x_i = 2500$			$\Sigma f_i d_i^2 = 16000$

$$\text{Mean} = \bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{2500}{100} = 25$$

$$\text{S.D.} = \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}} = \sqrt{\frac{16000}{100}}$$

$$\text{S.D.} = 12.649$$

$$\text{Variance} = (\text{S.D.})^2 = \sigma^2 = (12.649)^2$$

$$= 159.997$$

$$\text{Coefficient of variance} = \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{\sigma}{\bar{x}} \times 100$$

$$\text{Coefficient of variance} = \frac{12.649}{25} \times 100$$

$$= 50.596\ \%$$

2) If the mean is 82.5, standard deviation is 7.2, find the coefficient of variance.

Solution : Coefficient of variance $= \frac{\text{S.D.}}{\text{Mean}} \times 100$

$$= \frac{7.2}{82.5} \times 100$$

$$= 8.727\ \%$$

3) Coefficient of variation of a distribution is 75 % and standard deviation is 24.
What is its means

Solution : Coefficient of variation $= \frac{\text{S.D.}}{\text{Mean}} \times 100$

$$75 = \frac{24}{\text{Mean}} \times 100$$

$$\therefore \quad \text{Mean} = \frac{24}{75} \times 100$$

$$\text{Mean} = 32$$

Exercise:

1) The following tables gives the chest measurement of 30 men. Find variance and coefficient of variance.

Chest in cm	50-54	55-59	60-64	65-69	70-74	75-79	80-84	85 - 89	90-94
No. of men	3	2	3	2	1	4	9	5	1

2) Find variance of the data :

Class interval	0-30	30-60	60-90	90-120	120-150
Frequency	12	16	25	22	10

3) Find variance and the coefficient of variance for following data :

C.I.	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequency	9	17	43	82	81	44	24

4) Find mean and variance of the following

Age under	10	20	30	40	50	60	70	80
No. of person	15	30	53	75	100	110	115	125

5) If the coefficient of variation of certain data is 5 and mean is 60, find the standard deviation.

6) Coefficient of variation of certain distribution is 5 and mean is 60. Find the standard deviation.

7) If the mean of data is 12 and coefficient of variation of the data is 45%, then Find the standard deviation of the data.

8) The mean and variance of 10 observations are 4 and 9 respectively. Find their coefficient of variance

Comparison of Two Sets of Observations

Co-efficient of variance is the most important relative measure of dispersion .

Less the coefficient of variance the set is more consistent. If a set has greater coefficient of variance then it is not so reliable because it has less consistency, more variations.

Solved Examples:

1) Two sets of observation are given below

Set-I Set-II

$\bar{x} = 82.5$ $\bar{y} = 48.75$

$\sigma_x = 7.3$ $\sigma_y = 8.35$

Which of the two sets is more consistent ?

Solution :

$$\text{Coefficient of variance of set I} = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{7.3}{82.5} \times 100 = 8.848\ \%$$

Coefficient of variance of set II $= \frac{\sigma}{\bar{x}} \times 100$

$= \frac{8.35}{48.75} \times 100 = 17.12\ \%$

Set I is more consistent because having less coefficient of variance

2) In the two factories A and B engaged in the same industry, the average weekly wages and standard deviation are as follows :

Factories	Average wages	Standard deviation
A	34.5	5.0
B	28.5	4.5

Which factory is more consistent?

Solution :

Coefficient of variance of factory A $= \frac{\sigma}{\bar{x}} \times 100 = \frac{5.0}{34.5} \times 100$

$= 14.49\ \%$

Coefficient of variance of factory B $= \frac{\sigma}{\bar{x}} \times 100 = \frac{4.5}{28.5} \times 100$

$= 15.79\ \%$

$\therefore$ Factory A is more consistent because having less coefficient of variance

3) The data of run scored by two batsman A and B in five one day matches is given below :

Batsman	Average run scored	S.D.
A	44	5.1
B	54	6.31

State which batsman is more consistent

Solution : Coefficient of Variance of batsman A $= \frac{\sigma}{x} \times 100$

$= \frac{5.1}{44} \times 100$

$= 11.59\ \%$

Coefficient of Variance of batsman B $= \frac{\sigma}{x} \times 100$

$= \frac{6.31}{54} \times 100$

$= 11.68\ \%$

Coefficient of variance of Batsman A < Coefficient of variance of Batsman B.

Batsman A is more consistent.

Exercise:

1. The mean and the S.D. of two sets of variations are

	Set-I	**Set-II**
Mean	60	65
S.D.	5	6

which set is more consistent.

2. From the following data investigate which set is more consistent.

Set	**mean = $\bar{x}$**	**S.D. = σ**
Set I	83.4	5.9
Set II	51.85	7.45

3. In two factories A and B, engaged in the same area of the industry, the average weekly wages (in `) and the S.D. are as below :

Factory	**Average wages**	**S.D.**
A	34.5	5.0
B	28.5	4.5

which factory A or B has greater variability in individuals wages ?

4) An analysis of monthly wages paid to the workers in two firms A and B belonging to the same industry gives the following results :

	Firm-A	**Firm-B**
Average monthly wages (in `)	186	175
Variance of distribution of wages (in `)	81	100

In which firm is there greater variability?

5) The Mean of run scored by two batsmen A and B in a series of 10 innings are 50 and 12 respectively. The standard deviation of their runs is 15 and 2 respectively. Who is most consistent?

References used for learning manual

Web Resources:

a .www.scilab.org/ - SCI Lab

b. www.allmathcad.com/ - MathCAD

c. www.wolfram.com/mathematica/ - Mathematica

d. https://www.khanacademy.org/math?gclid=CNqHuabCys4CFdOJaAoddHoPig

Reference books:

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4. https://www.youtube.com/watch?v=kuixY2bCc_0
5. https://www.youtube.com/watch?v=UpUjJQGSlDY

Worksheet: